

# CONCENTRATION ON MINIMAL SUBMANIFOLDS FOR A YAMABE TYPE PROBLEM

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**ABSTRACT.** We construct solutions to a Yamabe type problem on a Riemannian manifold  $M$  without boundary and of dimension greater than 2, with nonlinearity close to higher critical Sobolev exponents. These solutions concentrate their mass around a non degenerate minimal submanifold of  $M$ , provided a certain geometric condition involving the sectional curvatures is satisfied. A connection with the solution of a class of P.D.E.'s on the submanifold with a singular term of attractive or repulsive type is established.

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let  $(M, g)$  be a compact Riemannian manifold of dimension  $m \geq 3$  without boundary. This paper deals with the semilinear elliptic problem

$$-\Delta_g u + hu = u^{q-1}, \quad u > 0, \quad \text{in } (M, g), \quad (1.1)$$

where the potential  $h \in C^2(M)$  is such that  $-\Delta_g + h$  is coercive and the exponent  $q > 2$ .

Existence of non-trivial solutions to problem (1.1) is strictly related to the position of  $q$  with respect to the critical Sobolev exponent  $2_m^* := \frac{2m}{m-2}$ . Indeed, in the subcritical case, i.e.  $q < 2_m^*$ , the Sobolev embedding  $H_g^1(M) \hookrightarrow L_g^q(M)$  is compact for any  $q \in (2, 2_m^*)$  and so

$$\inf_{\substack{u \in H_g^1(M) \\ u \neq 0}} \frac{\int_M (|\nabla_g u|^2 + hu^2) d\sigma_g}{\left( \int_M |u|^q d\sigma_g \right)^{2/q}} \quad (1.2)$$

is achieved and problem (1.1) has a non-trivial solution.

The critical case, i.e.  $q = 2_m^*$ , has important links with the well known Yamabe problem [25], namely find a metric  $\tilde{g}$  in the conformal class  $[g] = \{\phi g : \phi \in C^\infty(M), \phi > 0\}$  with constant scalar curvature  $\kappa$ . This is equivalent to set  $\tilde{g} = u^{4/(m-2)}g$  and to find a solution  $u$  to the Yamabe problem

$$-\Delta_g u + \frac{m-2}{4(m-1)} S_g u = \kappa u^{\frac{m+2}{m-2}}, \quad u > 0, \quad \text{in } (M, g), \quad (1.3)$$

where  $S_g$  is the scalar curvature of  $(M, g)$ . The Yamabe problem on the round sphere  $(\mathbb{S}^m, g_0)$ , equipped with the standard metric  $g_0$ , plays a crucial role in solving problem (1.3). Since the

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*Date:* September 26, 2014.

*2000 Mathematics Subject Classification.* 35B10, 35B33, 35J08, 58J05.

*Key words and phrases.* supercritical Yamabe type problem, concentration along minimal submanifolds, P.D.E.'s with attractive or repulsive type singularity.

<sup>0</sup>The first author was supported by Fondecyt grant 1130360 and Fondo Basal CMM. The second and the third authors have been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

scalar curvature of the round sphere is  $m(m-1)$  equation (1.3) reduces to

$$-\Delta_{g_0} u + \frac{m(m-2)}{4} u = u^{\frac{m+2}{m-2}}, \quad u > 0, \quad \text{in } (\mathbb{S}^m, g_0),$$

which is equivalent (via the stereographic projection) to the problem in the Euclidean space

$$-\Delta w = w^{\frac{m+2}{m-2}}, \quad w > 0, \quad \text{in } \mathbb{R}^m. \quad (1.4)$$

Problem (1.4) has infinitely many solutions (see [3]),

$$w_{\delta,y}(x) := \alpha_m \left( \frac{\delta}{\delta^2 + |x-y|^2} \right)^{\frac{m-2}{2}}, \quad x, y \in \mathbb{R}^m, \quad \delta > 0, \quad (1.5)$$

where  $\alpha_m := (m(m-2))^{\frac{m-2}{4}}$ . In the general case, Aubin in [1] proved that if

$$\mu_g(M, h) := \inf_{\substack{u \in H_g^1(M) \\ u \neq 0}} \frac{\int_M (|\nabla_g u|^2 + h u^2) d\sigma_g}{\left( \int_M |u|^{\frac{2m}{m-2}} d\sigma_g \right)^{\frac{m-2}{m}}}$$

is such that

$$\mu_g(M, h) < \mu_{g_0} \left( \mathbb{S}^m, \frac{m(m-2)}{4} \right) \quad (1.6)$$

then  $\mu_g(M, h)$  is achieved and so the corresponding critical problem (1.1) with  $q = 2_m^*$  has a non-trivial solution. The validity of (1.6) turns out to be strictly related to the position of the potential  $h$  with respect to the geometric potential

$$\omega(\xi) := \frac{m-2}{4(m-1)} S_g(\xi), \quad \xi \in M. \quad (1.7)$$

Indeed, if

$$h(\xi) < \omega(\xi) \quad \text{for any } \xi \in M, \quad (1.8)$$

it is not difficult to check that condition (1.6) holds. In the case of the Yamabe problem, i.e.  $h \equiv \omega$ , condition (1.6) is also true, but the proof is a delicate issue. It was proved by Trudinger [24] when  $\mu_g(M, \omega) \leq 0$ , by Aubin [1] when  $\mu_g(M, \omega) > 0$  and  $(M, g)$  is not locally conformally flat and  $m \geq 6$  and by Schoen [23] when  $\mu_g(M, \omega) > 0$  and either  $(M, g)$  is locally conformally flat or  $3 \leq m \leq 5$ .

We can summarize known results just saying that problem (1.1) has a non-trivial solution if either  $q < 2_m^*$  and no extra assumptions on  $h$ , or  $q = 2_m^*$  and  $h$  has to satisfy  $h \leq \omega$  on  $M$ . Therefore, it is natural to ask what happens when

$$h(\xi) > \omega(\xi) \quad \text{for some } \xi \in M \text{ or } p \text{ is supercritical, i.e. } p > 2_m^*.$$

A first partial answer was given by Micheletti, Pistoia and Vétois [21] in a perturbative setting. They consider the almost critical problem

$$-\Delta_g u + h u = u^{q_\epsilon - 1}, \quad u > 0, \quad \text{in } (M, g), \quad \text{with } q_\epsilon := 2_m^* \pm \epsilon \quad (1.9)$$

where  $\epsilon$  is a positive small parameter. If  $q_\epsilon = 2_m^* - \epsilon$  problem (1.9) is said to be slightly subcritical, while if  $q_\epsilon = 2_m^* + \epsilon$  it is said to be slightly supercritical. They proved the following result.

**Theorem 1.1.** [Theorem 1.1, Theorem 1.2, [21]] *Let  $m \geq 6$  and  $\xi_0 \in M$  be a non degenerate critical point of  $h - \omega$  (see (1.7)).*

- (i) If  $h(\xi_0) > \omega(\xi_0)$  then there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$  the slightly subcritical problem (1.9) with  $q_\epsilon = 2_m^* - \epsilon$  has a solution  $u_\epsilon$  such that

$$|\nabla u_\epsilon|^2 \rightharpoonup c_m \delta_{\xi_0} \text{ as } \epsilon \rightarrow 0.$$

- (ii) If  $h(\xi_0) < \omega(\xi_0)$  then there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$  the slightly supercritical problem (1.9) with  $q_\epsilon = 2_m^* + \epsilon$  has a solution  $u_\epsilon$  such that

$$|\nabla u_\epsilon|^2 \rightharpoonup c_m \delta_{\xi_0} \text{ as } \epsilon \rightarrow 0.$$

Here  $\delta_{\xi_0}$  stands for the Dirac measure supported on  $\xi_0$  and  $c_m$  is an explicit positive constant depending only on  $m$ .

The profile of  $u_\epsilon$  close to the concentration point  $\xi_0$  is given by (see (1.5))

$$u_\epsilon(x) \approx \alpha_m \left( \frac{\delta_\epsilon}{\delta_\epsilon^2 + |x - \xi_0|^2} \right)^{\frac{m-2}{2}}$$

where the concentration parameter  $\delta_\epsilon \sim d\sqrt{\epsilon}$ , as  $\epsilon \rightarrow 0$ , and the positive number  $d$  solves

$$a_m \underbrace{[h(\xi_0) - \omega(\xi_0)]}_{>0} d - \frac{b_m}{d} = 0 \quad \text{in the slightly sub-critical case} \quad (1.10)$$

or

$$a_m \underbrace{[h(\xi_0) - \omega S_g(\xi_0)]}_{<0} d + \frac{b_m}{d} = 0 \quad \text{in the slightly super-critical case.} \quad (1.11)$$

Here  $a_m$  and  $b_m$  are positive constants which only depend on  $m$ .

This result suggests to explore what happens when the exponent  $q$  is close to higher critical exponents. More precisely, for any integer  $0 \leq k \leq m-3$  we consider the  $(k+1)$ -st critical exponent  $2_{m,k}^* = \frac{2(m-k)}{m-k-2} = 2_{m-k,0}^*$ , namely the critical exponent for the Sobolev embedding  $H_h^1(N) \hookrightarrow L_h^q(N)$  where  $(N, h)$  is a  $(m-k)$ -dimensional Riemannian manifold. In particular,  $2_{m,0}^* = \frac{2m}{m-2}$  is the usual Sobolev critical exponent. We know by Theorem 1.1 that problem (1.1) when the exponent  $q$  approaches the first critical Sobolev exponent  $2_{m,0}^*$  has solutions which blow-up at a single point. A set consisting of a single point is a 0-dimensional submanifold of  $M$ . We ask: *if  $q$  approaches the  $(k+1)$ -st critical exponent  $2_{m,k}^*$ , do positive solutions blowing-up at  $k$ -dimensional submanifolds of  $M$  exist?*

Recently, a partial answer has been given by Dávila, Pistoia and Vaira in [4] when  $k=1$  and by Ghimenti, Micheletti and Pistoia in [15] in a symmetric setting. Here we deal with the general case. Let us consider the almost  $k$ -critical problem

$$-\Delta_g u + hu = u^{q_\epsilon-1}, \quad u > 0, \quad \text{in } (M, g), \quad \text{with } q_\epsilon := 2_{m,k}^* \pm \epsilon \quad (1.12)$$

where  $\epsilon$  is a positive small parameter. If  $q = 2_{m,k}^* - \epsilon$  problem (1.12) is said to be slightly  $k$ -th subcritical, while if  $q = 2_{m,k}^* + \epsilon$  it is said to be slightly  $k$ -th supercritical.

To state our result we need to introduce some geometric background. Let  $K \subset M$  be a  $k$ -dimensional submanifold. Set  $N := m-k$ . Let us introduce Fermi coordinates in  $\mathcal{M}$  near the submanifold  $K$ .

Let  $((E_a)_{a=1,\dots,k}, (E_i)_{i=1,\dots,N})$  be a local oriented and orthonormal frame field along  $K$ . At points  $\xi$  of  $K$ ,  $T_\xi \mathcal{M}$  splits as  $T_\xi K \oplus N_\xi K$ , where  $T_\xi K$  is the tangent bundle to  $K$  with orthonormal basis  $(E_a)_a$  and  $N_\xi K$  is the normal bundle, which is spanned by the orthonormal basis  $(E_j)_j$ . We assume that the normal vectors  $(E_i)_i$ ,  $i = 1, \dots, N$ , are parallel transported along  $K$ , namely

$$g(\nabla_{E_a} E_j, E_i) = 0 \quad \text{at } \xi, \text{ for any } i, j = 1, \dots, N, \quad a = 1, \dots, k. \quad (1.13)$$

Here  $\nabla$  is the connection associated with the metric  $g$ . We denote by  $\Gamma_a^b(\cdot)$  the 1-forms defined on the normal bundle of  $K$  by

$$\Gamma_{ai}^b := \Gamma_a^b(E_i) = g(\nabla_{E_a} E_b, E_i). \quad (1.14)$$

The *minimal condition on  $K$*  translates precisely into

$$\sum_{a=1}^k \Gamma_{ai}^a = 0 \text{ for any } i = 1, \dots, N. \quad (1.15)$$

In a neighborhood of  $\xi$  in  $K$ , we consider normal geodesic coordinates

$$f(y) := \exp_{\xi}^K \left( \sum_{a=1}^k y_a E_a \right), \quad y := (y_1, \dots, y_k), \quad (1.16)$$

where  $\exp^K$  is the exponential map on  $K$ . In a neighborhood of  $\xi$  in  $\mathcal{M}$ , we introduce

$$\mathfrak{F}(y, x) = \exp_{f(y)} \left( \sum_{i=1}^N x_i E_i \right); \quad (y, x) = ((y_a)_a, (x_i)_i), \quad (1.17)$$

where  $\exp_{f(y)}$  is the exponential map at  $f(y)$  in  $\mathcal{M}$ . It holds true that  $f(y) = \mathfrak{F}(y, 0) \in K$ . Let  $\tilde{g}_{ab}$  be the coefficients of the induced metric on  $K$  and let  $R_{\alpha\beta\gamma\delta}$  be the components of the curvature tensor computed at the point  $\xi$  of  $K$ . *Non-degeneracy of  $K$*  translates into the fact that the linear system

$$-\Delta_K \Phi_{\ell} + \sum_{m=1}^N \sum_{a,b=1}^k (\tilde{g}^{ab} R_{mab\ell} - \Gamma_{am}^b \Gamma_{b\ell}^a) \Phi_m = 0, \quad \ell = 1, \dots, N, \quad (1.18)$$

has only the trivial solution  $\Phi = (\Phi_1, \dots, \Phi_N) \equiv 0$ .

The Levi-Civita connection  $\nabla$  for  $g$  induces a connection  $\nabla^N$  on the normal bundle  $N_{\xi}K$ . We denote by

$$\mathcal{R}^N := \sum_{a=1}^k (R(E_a, \cdot) E_a)^N,$$

the curvature operator for this connection. The second fundamental form

$$B : T_{\xi}K \times T_{\xi}K \rightarrow N_{\xi}K, \quad B(X, Y) = (\nabla_X Y)^N$$

defines a symmetric operator  $\mathcal{B}^N := B^t \cdot B$ , in terms of the coefficients  $\Gamma_a^b := B(E_b, E_a)$ ,

$$g(\mathcal{B}^N X, Y) = \sum_{a,b=1}^k \Gamma_a^b(X) \Gamma_b^a(Y), \quad X, Y \in T_{\xi}M.$$

We also use the Ricci tensor

$$Ric(X, Y) = \sum_{j=1}^N g(R(X, E_j) E_j, Y) + \sum_{a=1}^k g(R(X, E_a) E_a, Y), \quad X, Y \in T_{\xi}M.$$

We introduce the quadratic form

$$\mathcal{Q}(X, Y) := \frac{1}{3} Ric(X, Y) - \frac{2}{3} g(\mathcal{R}^N X, Y) + g(\mathcal{B}^N X, Y), \quad X, Y \in N_{\xi}K.$$

Recall that  $N = m - k$ . We set

$$\Omega(\xi) := \frac{3(m-k-2)}{4(m-k-1)} \sum_{i=1}^{m-k} \mathcal{Q}(E_i, E_i), \quad \xi \in K. \quad (1.19)$$

The expression of  $\Omega$  in Fermi coordinates is given by

$$\hat{\Omega}(y) = \frac{3(N-2)}{4(N-1)} \left[ \sum_{i,j=1}^N \frac{1}{3} R_{jiji}(y) + \sum_{i=1}^N \sum_{a,b=1}^k (\tilde{g}^{ab} R_{iaib}(y) + \Gamma_{ai}^b(y) \Gamma_{bi}^a(y)) \right].$$

This function appears in our construction in (4.34). Observe that if  $k = 0$  the function  $\Omega$  is nothing but the geometric potential  $\frac{m-2}{4(m-1)} S_g(\xi)$  introduced in (1.7). Surprisingly enough, the function defined in (1.19) is not new in the literature, and it appears in a completely different context in [16] (Section 4). In fact, Mahmoudi, Mazzeo and Pacard in [16] deal with the existence of a family of Constant Mean Curvature submanifolds condensating to a fixed submanifold of a given Riemannian manifold. The limit submanifold has to be a closed non degenerate minimal submanifold. It would be interesting to further investigate the relation between our construction and the results in [16].

Existence of solutions to the almost  $k$ -th critical problem (1.12) is strictly related to the existence of solutions to the elliptic PDE's with a singularity of *attractive* type

$$-\Delta_K d + a_N \underbrace{[h(\xi) - \Omega(\xi)]}_{>0 \text{ in } K} d - \frac{b_N}{d} = 0, \quad d > 0 \text{ in } K \text{ in the slightly } k\text{-th sub-critical case} \quad (1.20)$$

or with a singularity of *repulsive* type

$$-\Delta_K d + a_N \underbrace{[h(\xi) - \Omega(\xi)]}_{<0 \text{ in } K} d + \frac{b_N}{d} = 0, \quad \mu > 0 \text{ in } K \text{ in the slightly } k\text{-th super-critical case.} \quad (1.21)$$

Here

$$a_N = \frac{4(N-1)}{(N-2)(N+2)} \text{ and } b_N = \frac{(N-2)^2(N-4)}{2(N+2)}. \quad (1.22)$$

This relation is new and unexpected. It seems to be the natural extension of conditions (1.10) and (1.11) to higher critical problems. More precisely, our main result reads as follows.

**Theorem 1.2.** *Let  $K \subset M$  be a closed non-degenerate minimal  $k$ -dimensional submanifold. Assume  $m - k \geq 7$ .*

- (i) *Assume (1.20) has a non-degenerate solution. Then there exists a sequence  $\epsilon = \epsilon_n \rightarrow 0$  such that the slightly  $k$ -th sub-critical problem (1.12) with  $q_\epsilon = 2_{m,k}^* - \epsilon$  has a solution  $u_\epsilon$  such that*

$$|\nabla u_\epsilon|^2 \rightharpoonup c_{m,k} \delta_K \text{ as } \epsilon \rightarrow 0.$$

- (ii) *Assume (1.21) has a non-degenerate solution. Then there exists a sequence  $\epsilon = \epsilon_n \rightarrow 0$  such that the slightly  $k$ -th super-critical problem (1.12) with  $q_\epsilon = 2_{m,k}^* + \epsilon$  has a solution  $u_\epsilon$  such that*

$$|\nabla u_\epsilon|^2 \rightharpoonup c_{m,k} \delta_K \text{ as } \epsilon \rightarrow 0.$$

Here  $\delta_K$  stands for the Dirac measure supported on  $K$  and  $c_{m,k}$  is an explicit positive constant depending only on  $m$  and  $k$ .

The profile of  $u_\epsilon$  close to the submanifold  $K$  is given in Fermi coordinate by (see (1.5))

$$u_\epsilon(y, x) \approx \alpha_{m-k} \left( \frac{\delta_\epsilon}{\delta_\epsilon^2 + |x|^2} \right)^{\frac{m-k-2}{2}}$$

where the concentration parameter  $\delta_\epsilon = \delta_\epsilon(y)$  satisfies  $\delta_\epsilon \sim d\sqrt{\epsilon}$  and the positive function  $d = d(y)$ , defined on  $K$ , solves either the attractive singular PDE (1.20) in the slightly sub-critical case or the repulsive singular PDE (1.21) in the slightly super-critical case.

It is important to point out that if  $\min_{\xi \in K} [h(\xi) - \Omega(\xi)] > 0$  then problem (1.20) has a non-degenerate solution as proved in Theorem 2.1. On the other hand, existence of solutions to problem (1.21) is a difficult issue, unless we deal with constant function  $h(\xi) - \Omega(\xi)$  (see Remark 2.6). Indeed, as far as we know, there is only one result in the literature, which was proved by del Pino, Manásevich and Montero in [8] in the case  $k = 1$ , when  $\max_{\xi \in K} [h(\xi) - \Omega(\xi)] < 0$  (see Theorem 2.5). We would like to stress the fact that existence of solutions to problem (1.21) is an interesting open question by itself, which as a by product allows to find solutions to the supercritical problem (1.12).

Another remark is that the result we find suggests that the natural extension to higher critical exponent of the classical Yamabe equation is

$$-\Delta_g u + \Omega(\xi)u = u^{\frac{m-k+2}{m-k-2}}, \quad u > 0 \text{ in } (M, g). \quad (1.23)$$

where  $\Omega$  is the function defined in (1.19). If  $k = 0$  Problem (1.23) reduces to the classical Yamabe equation since  $\frac{m-k+2}{m-k-2} = \frac{m+2}{m-2}$  and  $\Omega(\xi) = \frac{m-2}{4(m-1)}S_g(\xi)$ , as we already mentioned. A natural open question is thus: *does problem (1.23) have a solution?*

Finally, we point out some interesting problems, whose solutions could help in understanding equation (1.23).

- (i) Theorem 1.2 holds true when  $m \geq k + 7$ . The question is: *does problem (1.12) have any blowing-up solutions when  $3 \leq m \leq k + 6$ ?* The case  $k = 0$  was completely studied by Druet [11, 12].
- (ii) Theorem 1.2 holds true when  $h(\xi) \neq \Omega(\xi)$  for any  $\xi \in K$ . The question is: *does problem (1.12) have any blowing-up solutions if  $h(\xi) = \Omega(\xi)$  at some  $\xi \in K$ ?* The case  $k = 0$  was extensively studied by Esposito, Pistoia and Vetois [13] and by Esposito and Pistoia in [14].
- (iii) Theorem 1.2 holds true when  $q \rightarrow 2_{m,k}^*$ . The question is: *does problem (1.12) have any solutions if  $q = 2_{m,k}^*$ ?* The case  $k = 0$  is nothing but the well known Yamabe problem.

In the last few years several investigations have been carried out around the possibility of constructing singular limit solutions to non linear elliptic PDEs or problems in geometric analysis, depending on some parameters, whose mass or energy concentrate on sets of high dimension, like curves, surfaces, or higher dimensional sets. We refer the readers to [16, 7, 17, 18, 19, 20, 22, 5, 6, 9, 10] for instance, and the references therein. First contributions on concentration at higher dimensional set for problems involving higher critical Sobolev exponents are contained in the papers [7, 22, 6].

The general strategy used to prove all the above results is the so-called infinite dimensional version of the Liapunov-Schmidt reduction method. A main ingredient is to construct an approximate solution with arbitrary degree of accuracy in powers of  $\epsilon$ , in a neighborhood of the submanifold manifold  $K$ . This approximation is, at main order, a solution of some limit problem, which is independent of some of the variables. After this is done, one builds the desired solution by linearizing the equation around the approximation. The associated linear operator turns out to be invertible with inverse controlled in a suitable norm by certain large negative power of  $\epsilon$ , provided that  $\epsilon$  remains away from certain critical values where resonance occurs. The interplay

of the size of the error and that of the inverse of the linearization then makes it possible a fixed point scheme.

The rest of the paper is organized as follows.

We first discuss solvability and non-degeneracy of solutions to problems (1.20) and (1.21). This is done in Section 2. In Section 3 we introduce some scaled variables around the submanifold  $K$  and we describe the Laplace Beltrami operator in these new variables. Section 4 is devoted to the construction of the approximate solution to our problem using the local coordinates around the sub-manifold  $K$  introduced before. To perform this construction we need to invert a linear operator and to estimate the inverse. The proof of this result is postponed to Section 7. In Section 5 we define globally the approximation and we write the solution to our problem as the sum of the global approximation plus a remaining term. Thus we express our original problem as a non linear problem in the remaining term and we prove our Theorem. To solve such problem, we need to understand the invertibility properties of another linear operator. To do so we start expanding a quadratic functional associated to the linear problem. This is done in Section 6.

## 2. SOME REMARKS ON PDE'S WITH A SINGULAR TERM

First let us consider the attractive case, i.e. problem (1.20). We can deal with a more general situation.

**Theorem 2.1.** *Let  $(M, g)$  be a smooth Riemannian compact manifold without boundary. Assume  $\alpha, \beta \in C^0(M)$  and  $\min_M \alpha, \min_M \beta > 0$ . Then there exists a non-degenerate solution to*

$$\begin{cases} -\Delta_g u + \alpha u - \frac{\beta}{u} = 0 & \text{in } M \\ u > 0 & \text{in } M. \end{cases} \quad (2.1)$$

*Proof.* Let us prove that (2.1) does have a solution. Set  $L(u) := -\Delta_g u + \alpha u$ . Let us rewrite problem (2.1) in the following way

$$L(u) = f(x, u), \quad u > 0 \text{ in } M, \quad (2.2)$$

where  $L(u) = -\Delta_g u + \alpha u$  and  $f(x, u) := \frac{\beta}{u}$ . The linear operator  $L$  is coercive.

First of all, we prove that problem (2.1) has a lower solution  $\underline{u}$  and an upper solution  $\bar{u}$ , i.e.

$$L(\underline{u}) \leq f(x, \underline{u}) \text{ in } M \quad \text{and} \quad L(\bar{u}) \geq f(x, \bar{u}) \text{ in } M$$

such that

$$0 < \underline{u}(x) \leq \bar{u}(x) \text{ for any } x \in M.$$

It is enough to consider  $\underline{u}$  and  $\bar{u}$  as positive constant functions and to observe that  $L(c) - f(x, c) < 0$  if  $c$  is small enough and  $L(C) - f(x, C) > 0$  if  $C$  is large enough.

As a second step, we consider the modified problem

$$L(u) = \tilde{f}(x, u), \quad u > 0 \text{ in } M, \quad (2.3)$$

where

$$\tilde{f}(x, u) := \begin{cases} \frac{\beta(x)}{\underline{u}(x)} & \text{if } u(x) < \underline{u}(x) \\ \frac{\beta(x)}{u(x)} & \text{if } \underline{u}(x) \leq u(x) \leq \bar{u}(x) \\ \frac{\beta(x)}{\bar{u}(x)} & \text{if } u(x) > \bar{u}(x) \end{cases}$$

We point out that any solutions of the modified problem (2.3) is a solution to the problem (2.2). Indeed, assume  $u$  solves (2.3). We want to show that  $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$  for any  $x \in M$ .

Suppose, by contradiction that  $\max_M (\underline{u} - u) > 0$ . Then there exists a point  $x_0 \in M$  such that  $(\underline{u} - u)(x_0) := \max_M (\underline{u} - u) > 0$  and an open set  $\Gamma \subset M$  such that  $x_0 \in \Gamma$  and  $(\underline{u} - u)(x) \geq 0$  for any  $x \in \Gamma$ . Moreover, the function  $\underline{u} - u$  solves

$$L(\underline{u} - u) \leq 0 \text{ in } \Gamma.$$

Since it achieves a maximum at the point  $x_0$  which is in  $\Gamma$ , by remark 2.3 we immediately get a contradiction.

As a final step, we prove that problem (2.3) has a solution. We remark that  $u$  solves problem (2.3) if  $u$  is a fixed point of the operator  $K(u) := T(\tilde{f}(x, u))$ ,  $u \in C^0(M)$  where  $T$  is defined in Remark 2.4. By Remark 2.4 and by the definition of  $\tilde{f}(x, u)$  we deduce that  $K : C^0(M) \rightarrow C^0(M)$  is a compact operator and moreover that there exists  $R > 0$  such that  $\|K(u)\|_{C^0(M)} < R$  for any  $u \in C^0(M)$ . Hence for any  $t \in [0, 1]$ , using the homotopy invariance of the Leray-Schauder degree, we get

$$\deg(I - K, B(0, R)) = \deg(I - tK, B(0, R)) = \deg(I, B(0, R)) = 1$$

and so problem (2.3) has a solution. In order to prove that it is non degenerate, we point out that the linearized equation

$$-\Delta_g v + \alpha v + \frac{\beta}{u^2} v = 0, \text{ in } M.$$

has only the trivial solution, since  $\alpha$  and  $\beta$  are strictly positive functions on  $M$ .

That concludes the proof.  $\square$

**Remark 2.2.**  $L$  satisfies the maximum principle, namely if  $u \in H^1(M)$  is such that  $Lu \leq 0$  in  $M$  then  $u \leq 0$  in  $M$ .

**Remark 2.3.** Assume that for some open set  $\Gamma \subset M$  the function  $u \in H^1(M)$  solves  $Lu \leq 0$  in  $\Gamma$ . Then if  $u$  achieves its maximum at a point  $x \in \Gamma$  then  $u(x) \leq 0$ .

**Remark 2.4.** For any  $h \in C^0(M)$  there exists a unique  $w \in C^2(M)$  such that  $Lu = h$  in  $M$ . The linear map  $T : C^0(M) \rightarrow C^0(M)$  defined by  $Th = w$  is continuous and compact.

*Proof.* It is enough to remark that the linear map  $T : C^0(M) \rightarrow C^2(M)$  is continuous, because by standard elliptic regularity theory there exists a constant  $c$  which only depends on  $M$  and  $g$  such that

$$\|Th\|_{C^2(M)} \leq c\|h\|_{C^0(M)} \text{ for any } h \in C^0(M).$$

Moreover, the embedding  $C^2(M) \hookrightarrow C^0(M)$  is compact because of the Ascoli-Arzelá Theorem.  $\square$

As far as it concerns the repulsive case, i.e. equation (1.21), we quote the results obtained by del Pino, Manásevich and Montero in [8] in the case  $k = 1$  and by Dávila, Pistoia and Vaira in [4]. Let us consider the more general problem

$$\begin{cases} -\Delta_g u + \alpha u + \frac{\beta}{u} = 0 & \text{in } M \\ u > 0 & \text{in } M, \end{cases} \quad (2.4)$$

where  $\alpha, \beta \in C^0(M)$ , with  $\min_M \beta > 0$ .

**Theorem 2.5.** Let  $M$  be a one-dimensional manifold whose length is  $\ell$ . Assume

$$-\left(\frac{(\kappa + 1)\pi}{2\ell}\right)^2 < \max_{\xi \in M} \alpha(\xi) < -\left(\frac{\kappa\pi}{2\ell}\right)^2 < 0, \quad (2.5)$$



for some integer  $\kappa \geq 1$ . Then problem (2.4) has a solution (see [8]). Moreover, it is non-degenerate for most functions  $\alpha$ 's (see [4]).

In the general case, we can only make a few remarks.

**Remark 2.6.** (i) If (2.4) has a solution, then  $\min_{\xi \in M} \alpha(\xi) < 0$ .  
(ii) Let  $\alpha = a$  and  $\beta = b$  be constants. If  $a < 0$ , then problem (2.4) has a constant solution, which is non-degenerate if in addition  $-\lambda_{\kappa+1} < 2a < -\lambda_{\kappa} < 0$  holds for some  $\kappa$ . Here  $(\lambda_{\kappa})_{\kappa \geq 1}$  denotes the sequence of eigenvalues of  $-\Delta_M u = \lambda_{\kappa} u$  on  $M$ .

*Proof.* To prove (i) it is enough to integrate equation (2.4) on  $M$ , so we get

$$\int_M \alpha(\xi) u(\xi) d\xi + \int_M \frac{\beta(\xi)}{u(\xi)} d\xi = 0$$

which implies that  $\alpha$  has to be negative somewhere in  $M$ . The proof of (ii) follows by straightforward computations.  $\square$

It would be really interesting to find conditions on  $\alpha$  and  $\beta$  which ensure the existence of a solution to problem (2.4) in a more general setting.

### 3. LAPLACE-BELTRAMI OPERATOR IN SCALED VARIABLES

In this section we describe the Laplace-Beltrami operator in some scaled variables, by means of the Fermi coordinates introduced in (1.17).

Let  $\mu_{\epsilon}$  be a positive smooth function  $\mu_{\epsilon} = \mu_{\epsilon}(y)$  defined on  $K$  which we assume to be uniformly bounded, as  $\epsilon \rightarrow 0$ , along  $K$ . Let also  $\Phi_{\epsilon}$  be a smooth normal section (in  $M$ )  $\Phi_{\epsilon} : K \rightarrow NK$  defined by  $\Phi_{\epsilon}(y) = \Phi_{\epsilon}^j(y) E_j$ , and we assume that  $\Phi_{\epsilon}^j(y)$ ,  $j = 1, \dots, N$ , are functions uniformly bounded, as  $\epsilon \rightarrow 0$ , in  $K$ . Having introduced the above function, we define the following change of variables

$$u(\mathfrak{F}(y, x)) = (1 + \alpha_{\epsilon})(\sqrt{\epsilon} \mu_{\epsilon}(y))^{-\frac{N-2}{2}} v\left(\frac{y}{\sqrt{\epsilon}}, \frac{x - \epsilon \Phi_{\epsilon}(y)}{\sqrt{\epsilon} \mu_{\epsilon}(y)}\right), \quad (3.1)$$

where  $\mathfrak{F}(y, x)$  is the change of variables defined in (1.17). and

$$v = v(z, \xi), \quad z = \frac{y}{\sqrt{\epsilon}}, \quad \xi = \frac{x - \epsilon \Phi_{\epsilon}}{\sqrt{\epsilon} \mu_{\epsilon}}. \quad (3.2)$$

In (3.1)  $\alpha_{\epsilon}$  is a number defined so that  $(1 + \alpha_{\epsilon})^{p \pm \epsilon - 1} \epsilon^{\mp \frac{N-2}{4} \epsilon} = 1$ , that is,

$$\alpha_{\epsilon} = \epsilon^{\pm \frac{(N-2)^2}{16 \pm 4(N-2)\epsilon} \epsilon} - 1. \quad (3.3)$$

To emphasize the dependence of the above change of variables on  $\mu_{\epsilon}$  and  $\Phi_{\epsilon}$ , we will use the notation

$$u = \mathcal{T}_{\mu_{\epsilon}, \Phi_{\epsilon}}(v) \iff u \text{ and } v \text{ satisfy (3.1)}. \quad (3.4)$$

Recall that the original variables  $(y, x) \in \mathbb{R}^{k+N}$  are *local* coordinates along  $K$ . Thus we let the variables  $(z, \xi)$  vary in the set  $\mathcal{D}$  defined by

$$\mathcal{D} = \left\{ (z, \xi) : \sqrt{\epsilon} z \in K, \quad |\xi| < \frac{\eta}{\sqrt{\epsilon}} \right\} \quad (3.5)$$

for some small and fixed positive number  $\eta$  that will be fixed in the sequel. We will also use the notation  $\mathcal{D} = K_\epsilon \times \hat{\mathcal{D}}$ , where  $K_\epsilon = \frac{K}{\sqrt{\epsilon}}$  and

$$\hat{\mathcal{D}} = \left\{ \xi : |\xi| < \frac{\eta}{\sqrt{\epsilon}} \right\}. \quad (3.6)$$

We note that  $\partial\hat{\mathcal{D}} = \left\{ \xi \in \hat{\mathcal{D}} : |\xi| = \frac{\eta}{\sqrt{\epsilon}} \right\}$ .

We are interested in computing the Laplace Beltrami operator in the new variables  $(z, \xi)$  in terms of the parameter  $\epsilon$ , of the function  $\mu_\epsilon(y)$  and of the normal section  $\Phi_\epsilon$ .

We have the validity of the following

**Lemma 3.1.** *Given the change of variables defined in (3.1), the following expansion for the Laplace Beltrami operator holds true*

$$(1 + \alpha_\epsilon)^{-1} \epsilon^{\frac{N+2}{4}} \mu_\epsilon^{\frac{N+2}{2}} \Delta_g u = \mathcal{A}_{\mu_\epsilon, \Phi_\epsilon}(v) := \mu_\epsilon^2 \Delta_{K_\epsilon} v + \Delta_\xi v + \sum_{\ell=0}^2 \mathcal{A}_\ell v + B(v). \quad (3.7)$$

Above, the expression  $\mathcal{A}_k$  denotes specific differential operators, respectively defined as follows

$$\begin{aligned} \mathcal{A}_0 v &= -\epsilon \mu_\epsilon \Delta_K \mu_\epsilon (\gamma v + D_\xi v [\xi]) \\ &+ \epsilon |\nabla_K \mu_\epsilon|^2 [D_\xi v [\xi]^2 + 2(1 + \gamma) D_\xi v [\xi] + \gamma(1 + \gamma) v] \\ &- \epsilon^{\frac{3}{2}} \mu_\epsilon D_\xi v [\Delta_K \Phi_\epsilon] + \epsilon^{\frac{3}{2}} \nabla_K \mu_\epsilon \cdot \{2D_\xi v [\xi] + ND_\xi v\} [\nabla_K \Phi_\epsilon] \\ &+ \epsilon^2 D_\xi v [\nabla_K \Phi_\epsilon]^2 \\ &- 2\epsilon^2 \mu_\epsilon g^{ab} [D_\xi(\partial_a v) [\partial_b \mu_\epsilon \xi] + \epsilon^{\frac{1}{2}} D_\xi(\partial_a v) [\partial_b \Phi_\epsilon] + \gamma \partial_a \mu_\epsilon \partial_b v], \end{aligned} \quad (3.8)$$

where we have set  $\gamma = \frac{N-2}{2}$ ,

$$\mathcal{A}_1 v = -\frac{\epsilon}{3} \sum_{i,j} \left[ \sum_{m,l} R_{mijl} (\mu_\epsilon \xi_m + \sqrt{\epsilon} \Phi_\epsilon^m) (\mu_\epsilon \xi_l + \sqrt{\epsilon} \Phi_\epsilon^l) \right] \partial_{ij}^2 v, \quad (3.9)$$

and

$$\mathcal{A}_2 v = \epsilon \mu_\epsilon \sum_j \left[ \sum_s \frac{2}{3} R_{mssj} + \sum_{m,a,b} (\tilde{g}^{ab} R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \right] (\mu_\epsilon \xi_m + \sqrt{\epsilon} \Phi_\epsilon^m) \partial_j v. \quad (3.10)$$

Finally, the operator  $B(v)$  can be described as follows:  $B(v) = \epsilon^2 \hat{B}(v)$

$$\begin{aligned} \hat{B}(v) &= O(|\mu_\epsilon \xi + \sqrt{\epsilon} \Phi|^3) \partial_{ij}^2 v \\ &+ O(|\mu_\epsilon + \sqrt{\epsilon} \Phi_\epsilon + \partial_{z_a}(\mu_\epsilon + \sqrt{\epsilon} \Phi_\epsilon)|^2) O(v + \xi_i \partial_{\xi_i} v). \end{aligned}$$

We recall that the symbols  $\partial_a$ ,  $\partial_{\bar{a}}$  and  $\partial_i$  denote the derivatives with respect to  $\partial_{y_a}$ ,  $\partial_{z_a}$  and  $\partial_{\xi_i}$  respectively.

*Proof.* Recall that the Laplace-Beltrami operator is defined by

$$\Delta_g = \frac{1}{\sqrt{\det g}} \partial_A (\sqrt{\det g} (g)^{AB} \partial_B),$$

where indices  $A$  and  $B$  run between 1 and  $n = N + k$ . In other words

$$\Delta_g = (g)^{AB} \partial_{AB}^2 + \partial_A (g)^{AB} \partial_B + \partial_A (\log \sqrt{\det g}) (g)^{AB} \partial_B \quad (3.11)$$

If now  $u$  and  $v$  are defined as in (3.1), we have

$$(1 + \alpha_\epsilon)^{-1} \epsilon^{\frac{N}{4}} \mu_\epsilon^{\frac{N}{2}} \partial_{x_j} u = \partial_{\xi_j} v, \quad (1 + \alpha_\epsilon)^{-1} \epsilon^{\frac{N+2}{4}} \mu_\epsilon^{\frac{N+2}{2}} \partial_{x_j, x_i}^2 u = \partial_{\xi_j, \xi_i}^2 v$$

and

$$\begin{aligned} (1 + \alpha_\epsilon)^{-1} \epsilon^{\frac{N+2}{4}} \mu_\epsilon^{\frac{N+2}{2}} \partial_{y_a, y_b}^2 u &= \mu_\epsilon^2 \partial_{z_a z_b}^2 v + \mu_\epsilon^{\frac{N-2}{2}} \partial_{z_a z_b}^2 (\mu_\epsilon^{-\frac{N-2}{2}}) v \\ &+ 2 \mu_\epsilon^{\frac{N-2}{2}} \partial_{z_a}^2 (\mu_\epsilon^{-\frac{N-2}{2}}) [\partial_{z_b} v + \partial_{z_b} (\mu_\epsilon^{-1}) \nabla v \cdot \xi - \sqrt{\epsilon} \nabla v \cdot \partial_{z_b} \Phi_\epsilon] \\ &+ \partial_{z_a} [\partial_{z_b} (\mu_\epsilon^{-1}) \nabla v \cdot \xi - \sqrt{\epsilon} \nabla v \cdot \partial_{z_b} \Phi_\epsilon] \end{aligned}$$

On the other hand, by our choice of coordinates (1.16), on  $K$  the metric  $g$  splits in the following way

$$g(q) = g_{ab}(q) dy_a \otimes dy_b + g_{ij}(q) dx_i \otimes dx_j, \quad q \in K. \quad (3.12)$$

If we denote by  $r$  the distance function from  $K$ , at any point  $\mathfrak{F}(y, x)$  (see (1.17), we have

$$\begin{aligned} g_{ij}(y, x) &= \delta_{ij} + \frac{1}{3} R_{istj} x_s x_t + \mathcal{O}(r^3); \\ g_{aj}(y, x) &= \mathcal{O}(r^2); \\ g_{ab}(y, x) &= \tilde{g}_{ab} - [\tilde{g}_{ac} \Gamma_{bi}^c + \tilde{g}_{bc} \Gamma_{ai}^c] x_i + [R_{sabl} + \tilde{g}_{cd} \Gamma_{as}^c \Gamma_{dl}^b] x_s x_l + \mathcal{O}(r^3). \end{aligned}$$

Here  $a = 1, \dots, k$ ,  $i, j = 1, \dots, N$ . See [6]. Let now  $g^\epsilon$  be the scaled metric on  $\mathcal{M}_\epsilon = \epsilon^{-1/2} \mathcal{M}$ , whose coefficients are defined by

$$g_{\alpha, \beta}^\epsilon(z, x) = g_{\alpha, \beta}(\sqrt{\epsilon} z, \sqrt{\epsilon} x).$$

For the metric  $g^\epsilon$  in the above coordinates  $(z, x)$  we have the expansions

$$\begin{aligned} g_{ij}^\epsilon &= \delta_{ij} + \frac{\epsilon}{3} R_{istj} x_s x_t + \mathcal{O}(\epsilon^{\frac{3}{2}}(|x|^3)), \quad 1 \leq i, j \leq N; \\ g_{aj}^\epsilon &= \mathcal{O}(\epsilon|x|^2) \quad 1 \leq a \leq k, 1 \leq j \leq N; \\ g_{ab}^\epsilon &= \tilde{g}_{ab}^\epsilon - \sqrt{\epsilon} \left\{ \tilde{g}_{ac}^\epsilon \Gamma_{bi}^c + \tilde{g}_{bc}^\epsilon \Gamma_{ai}^c \right\} x_i + \epsilon \left[ R_{sabl} + \tilde{g}_{cd}^\epsilon \Gamma_{as}^c \Gamma_{dl}^b \right] x_s x_l + \mathcal{O}(\epsilon^{\frac{3}{2}}|x|^3), \\ &1 \leq a, b \leq k. \end{aligned}$$

Thus we first conclude that

$$\begin{aligned} (1 + \alpha_\epsilon)^{-1} \epsilon^{\frac{N+2}{4}} \mu_\epsilon^{\frac{N+2}{2}} (g)^{AB} \partial_{AB}^2 &= \mu_\epsilon^2 g^{ab} \partial_{z_a z_b}^2 v + \partial_{jj}^2 v \\ &- \frac{\epsilon}{3} R_{mijl} (\mu_\epsilon \xi_l + \sqrt{\epsilon} \Phi_\epsilon^l) (\mu_\epsilon \xi_m + \sqrt{\epsilon} \Phi_\epsilon^m) \partial_{\xi_i \xi_j} v \\ &+ \mathcal{B}_0(v) + \mathcal{B}_1(v) \end{aligned} \quad (3.13)$$

where

$$\begin{aligned}
\mathcal{B}_0(v) &= -\epsilon \mu_\epsilon g^{ab} \partial_{z_a z_b}^2 \mu_\epsilon \left( \frac{N-2}{2} v + D_\xi v [\xi] \right) \\
&+ \epsilon g^{ab} \partial_{z_a} \mu_\epsilon \partial_{z_b} \mu_\epsilon \left[ D_{\xi\xi} v [\xi]^2 + N D_\xi v [\xi] + \frac{(N-2)N}{4} v \right] \\
&- \epsilon^{\frac{3}{2}} \mu_\epsilon g^{ab} \partial_{z_a z_b}^2 \Phi_\epsilon^j \partial_{\xi_j} v + \epsilon^{\frac{3}{2}} g^{ab} \partial_{z_a} \mu_\epsilon \partial_{z_b} \Phi_\epsilon^l \left\{ 2 \partial_{\xi_j \xi_l}^2 v \xi_j + N \partial_{\xi_l} v \right\} \\
&+ \epsilon^2 g^{ab} \partial_{z_a} \Phi_\epsilon^j \partial_{z_b} \Phi_\epsilon^l \partial_{\xi_j \xi_l}^2 v \\
&- 2 \epsilon^2 \mu_\epsilon g^{ab} \left[ D_\xi (\partial_{\bar{a}} v) [\partial_b \mu_\epsilon \xi] + \epsilon^{\frac{1}{2}} D_\xi (\partial_{\bar{a}} v) [\partial_b \Phi_\epsilon] + \gamma \partial_a \mu_\epsilon \partial_{\bar{b}} v \right],
\end{aligned}$$

and

$$\mathcal{B}_1(v) = O(\epsilon^2 |\mu_\epsilon \xi + \sqrt{\epsilon} \Phi_\epsilon|^3) \partial_{\xi_i \xi_j}^2 v.$$

Moreover

$$\begin{aligned}
(1 + \alpha_\epsilon)^{-1} \epsilon^{\frac{N+2}{4}} \mu_\epsilon^{\frac{N+2}{2}} \partial_A (g)^{AB} \partial_B u &= \mu_\epsilon^2 \partial_a (g^{ab}) \partial_{z_b} v \\
&+ \frac{\epsilon}{3} \mu_\epsilon R_{liij} (\mu_\epsilon \xi_l + \sqrt{\epsilon} \Phi_\epsilon^l) \partial_{\xi_j} v \\
&+ \mathcal{B}_2(v)
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\mathcal{B}_2(v) &= -\epsilon \mu_\epsilon \partial_{z_a} (g^{ab}) \partial_{z_b} \mu_\epsilon \left( \frac{N-2}{2} v + D_\xi v [\xi] \right) \\
&- \epsilon^{\frac{3}{2}} \mu_\epsilon \partial_{z_a} (g^{ab}) \partial_{z_b} \Phi_\epsilon^j \partial_{\xi_j} v \\
&+ \epsilon^2 O(|\mu_\epsilon + \sqrt{\epsilon} \Phi_\epsilon + \partial_{z_a} (\mu_\epsilon + \sqrt{\epsilon} \Phi_\epsilon)|^2) O(v + \xi_i \partial_{\xi_i} v).
\end{aligned}$$

Finally, we recall that we have the validity of the following expansions for the square root of the determinant of  $g^\epsilon$  and the log of determinant of  $g^\epsilon$

$$\begin{aligned}
\sqrt{\det g^\epsilon} &= \sqrt{\det g^\epsilon} \times \\
&\left\{ 1 + \frac{\epsilon}{6} R_{miil} x_m x_l + \frac{\epsilon}{2} \left( \tilde{g}^{ab} R_{mabl} - \Gamma_{am}^c \Gamma_{cl}^a \right) x_m x_l + \epsilon^{\frac{3}{2}} \mathcal{O}(|x|^3) \right\}
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\log (\det g^\epsilon) &= \log (\det g^\epsilon) + \frac{\epsilon}{3} R_{miil} x_m x_l \\
&+ \epsilon \left( \tilde{g}^{ab} R_{mabl} - \Gamma_{am}^c \Gamma_{cl}^a \right) x_m x_l + \mathcal{O}(\epsilon^{\frac{3}{2}} |x|^3).
\end{aligned}$$

See for instance [6].

So we get

$$\begin{aligned}
(1 + \alpha_\epsilon)^{-1} \epsilon^{\frac{N+2}{4}} \mu_\epsilon^{\frac{N+2}{2}} \partial_A (\log \sqrt{\det g}) (g)^{AB} \partial_B u &= \partial_a (\log \sqrt{\det g}) (g)^{ab} \partial_b v \\
&+ \epsilon \left( \frac{R_{mssj}}{3} + (\tilde{g}^{ab} R_{mabj} - \Gamma_{am}^c \Gamma_{cj}^a) \right) (\mu_\epsilon \xi_m + \sqrt{\epsilon} \Phi_\epsilon^m) \partial_{\xi_j} v \\
&+ \mathcal{B}_3(v).
\end{aligned} \tag{3.16}$$

Here  $\mathcal{B}_3(v)$  is a function that can be described as follows

$$\mathcal{B}_3(v) = \epsilon^2 O(|\mu_\epsilon + \sqrt{\epsilon}\Phi_\epsilon + \partial_{z_a}(\mu_\epsilon + \sqrt{\epsilon}\Phi_\epsilon)|^2) O(v + \xi_i \partial_{\xi_i} v).$$

Colleting (3.13), (3.14) and (3.16) in (3.11), we get the proof of the Lemma.  $\square$

#### 4. CONSTRUCTION OF AN APPROXIMATE SOLUTION

Using the local coordinates along the submanifold  $K$  introduced in Section 3, after performing the change of variables in (3.1), the original equation in  $u$  reduces locally close to  $K_\epsilon$  to the following equation in  $v$

$$-\mathcal{A}_{\mu_\epsilon, \Phi_\epsilon} v + \epsilon \mu_\epsilon^2 h v - \mu_\epsilon^{\mp \epsilon \frac{N-2}{2}} v^{p \pm \epsilon} = 0, \quad (4.1)$$

where  $\mathcal{A}_{\mu_\epsilon, \Phi_\epsilon}$  is defined in (3.7) and  $p = \frac{N+2}{N-2}$ . Let us denote by  $\Xi_\epsilon$  the operator given by

$$\Xi_\epsilon(v) := -\mathcal{A}_{\mu_\epsilon, \Phi_\epsilon} v + \epsilon \mu_\epsilon^2 h v - \mu_\epsilon^{\mp \epsilon \frac{N-2}{2}} v^{p \pm \epsilon}. \quad (4.2)$$

This section is devoted to build an approximate solution to Problem (4.1) locally around  $K_\epsilon$ , in the set  $\mathcal{D} = K_\epsilon \times \hat{\mathcal{D}}$ , (see Section 3).

Let  $r$  be an integer. For a function  $w$  defined in  $\mathcal{D} = K_\epsilon \times \hat{\mathcal{D}}$ , we define

$$\|w\|_{\epsilon, r} := \sup_{(z, \xi) \in K_\epsilon \times \hat{\mathcal{D}}} \left( (1 + |\xi|^2)^{\frac{r}{2}} |w(z, \xi)| \right). \quad (4.3)$$

Let  $\sigma \in (0, 1)$ . We define

$$\|w\|_{\epsilon, r, \sigma} := \|w\|_{\epsilon, r} + \sup_{(z, \xi) \in K_\epsilon \times \hat{\mathcal{D}}} \left( (1 + |\xi|^2)^{\frac{r+\sigma}{2}} [w]_{\sigma, B(\xi, 1)} \right) \quad (4.4)$$

where we have denoted

$$[w]_{\sigma, B(\xi, 1)} := \sup_{\xi_1, \xi_2 \in B(\xi, 1)} \frac{|w(z, \xi_2) - w(z, \xi_1)|}{|\xi_1 - \xi_2|^\sigma} \quad (4.5)$$

The main result of this section is as follows.

**Lemma 4.1.** *There exist  $\epsilon_0 > 0$ ,  $\eta > 0$  in the definition of  $\mathcal{D}$  in (3.5), and a constant  $C > 0$ , such that, for any integer  $I$  and for all  $\epsilon \in (0, \epsilon_0)$  there exist a smooth function  $\mu_{I+1, \epsilon} : K \rightarrow \mathbb{R}$ , a smooth normal section  $\Phi_{I+1, \epsilon} : K \rightarrow NK$ , of the form  $\Phi_{I+1, \epsilon}(y) = \Phi_{I+1, \epsilon}^j(y) E_j$*

$$\|\mu_{I+1, \epsilon}\|_\infty + \|\partial_a \mu_{I+1, \epsilon}\|_\infty + \|\partial_a^2 \mu_{I+1, \epsilon}\|_\infty \leq C \quad (4.6)$$

$$\|\Phi_{I+1, \epsilon}\|_\infty + \|\partial_a \Phi_{I+1, \epsilon}\|_\infty + \|\partial_a^2 \Phi_{I+1, \epsilon}\|_\infty \leq C, \quad (4.7)$$

and a positive function  $v_{I+1, \epsilon} : K_\epsilon \times \hat{\mathcal{D}} \rightarrow \mathbb{R}$  such that

$$-\mathcal{A}_{\mu_{I+1, \epsilon}, \Phi_{I+1, \epsilon}}(v_{I+1, \epsilon}) + \epsilon \mu_{I+1, \epsilon}^2 h v_{I+1, \epsilon} - \mu_{I+1, \epsilon}^{\mp \epsilon \frac{N-2}{2}} v_{I+1, \epsilon}^{p \pm \epsilon} = \mathcal{E}_{I+1, \epsilon} \quad \text{in } \mathcal{D}$$

with

$$\|v_{I+1, \epsilon} - v_{I, \epsilon}\|_{\epsilon, N-4, \sigma} \leq C \epsilon^{I+\frac{1}{2}} \quad (4.8)$$

and

$$\|\mathcal{E}_{I+1, \epsilon}\|_{\epsilon, N-2, \sigma} \leq C \epsilon^{I+\frac{1}{2}}. \quad (4.9)$$

We refer to (3.7) for the definition of  $\mathcal{A}_{\mu_\epsilon, \Phi_\epsilon}$ , to (3.5) for  $K_\epsilon \times \hat{\mathcal{D}}$ .

The proof of Lemma 4.1 is based on an explicit construction of the functions  $\mu_{\epsilon, I+1}$ ,  $\Phi_{\epsilon, I+1}$  and  $v_{\epsilon, I+1}$ , via an iterative scheme, in the spirit developed in [6]. Fix an integer  $I > 1$ , we will define the functions  $\mu_{\epsilon, I}$  and  $\Phi_{\epsilon, I}$  respectively of the form

$$\mu_{I, \epsilon} := \mu_0 + \epsilon \mu_1 + \epsilon^2 \mu_2 + \dots + \epsilon^{I-1} \mu_{I-1}, \quad (4.10)$$

and

$$\Phi_{\epsilon} := \Phi_{1, \epsilon} + \Phi_{2, \epsilon} + \dots + \Phi_{I-1, \epsilon}. \quad (4.11)$$

to be solutions of certain linear elliptic PDEs on the sub manifold  $K$ . The solvability of these equations is related to the result contained in Section 2. At each step  $I$ , we also define

$$v_I(z, \xi) := w_0(\xi) + w_{1, \epsilon}(z, \xi) + w_{2, \epsilon}(z, \xi) + w_{3, \epsilon}(z, \xi) + \dots + w_{I, \epsilon}(z, \xi), \quad (4.12)$$

where each term  $w_{j, \epsilon}$  will also be solution of a linear problem, this time defined on  $\mathcal{D}$ . The function  $w_0$  has been already defined as solution to

$$\Delta u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N, \quad (4.13)$$

given explicitly by

$$w_0(\xi) = \alpha_N (1 + |\xi|^2)^{-\frac{N-2}{2}}. \quad (4.14)$$

We consider the domain  $\mathcal{D}$  defined as (3.5) and for function  $\phi$  defined on  $\mathcal{D}$ , an operator of the form

$$L(\phi) := -\Delta_{\xi} \phi - p w_0^{p-1} \phi + \epsilon a(\epsilon z) \phi,$$

where  $a$  is a given smooth function  $a : K \rightarrow \mathbb{R}$  with  $a(y) \geq \lambda > 0$  for all  $y \in K$ .

Let us introduce the functions

$$Z_j(\xi) = \frac{\partial w_0}{\partial \xi_j}, \quad j = 1, \dots, N \quad \text{and} \quad Z_0(\xi) = \xi \cdot \nabla w_0(\xi) + \frac{N-2}{2} w_0(\xi) \quad (4.15)$$

that are known to be the only bounded solutions to the linearized equation around  $w_0$  of problem (4.13)

$$-\Delta \phi - p w_0^{p-1} \phi = 0 \quad \text{in } \mathbb{R}^N.$$

See [2].

Given a function  $g : K \times \hat{\mathcal{D}} \rightarrow \mathbb{R}$  that depends smoothly on the variable  $y \in K$ , we want to find a linear theory for the following linear problem

$$\begin{cases} L(\phi) = h, & \text{in } \mathcal{D} \\ \phi = 0 & \text{on } \partial \hat{\mathcal{D}} \\ \int_{\hat{\mathcal{D}}} \phi(\epsilon z, \xi) Z_j(\xi) d\xi = 0 & \forall z \in K_{\epsilon}, \quad j = 0, \dots, N. \end{cases} \quad (4.16)$$

We have the validity of the following result.

**Proposition 4.2.** *Let  $r$  be an integer such that  $4 < r < N$ .*

*Let  $a : K \rightarrow \mathbb{R}$  be a smooth function, such that  $a(y) \geq \lambda > 0$  for all  $y \in K$ . Then there exist  $\epsilon_0 > 0$ ,  $\eta > 0$ , that depends only on  $\sup_{y \in K} |a(y)|$ , in the definition of  $\mathcal{D}$  in (3.5), and  $C > 0$  such that, for any  $\epsilon \in (0, \epsilon_0)$  and for any function  $h : K \times \hat{\mathcal{D}} \rightarrow \mathbb{R}$  that depends smoothly on the variable  $y \in K$ , such that  $\|h\|_{\epsilon, r}$  is bounded, uniformly in  $\epsilon$ , and*

$$\int_{\hat{\mathcal{D}}} h(\epsilon z, \xi) Z_j(\xi) d\xi = 0 \quad \text{for all } z \in K_{\epsilon}, \quad j = 0, 1, \dots, N,$$

*then there exists a solution  $\phi$  of problem (4.16) such that*

$$\|D_{\xi}^2 \phi\|_{\epsilon, r, \sigma} + \|D_{\xi} \phi\|_{\epsilon, r-1, \sigma} + \|\phi\|_{\epsilon, r-2, \sigma} \leq C \|h\|_{\epsilon, r, \sigma} \quad (4.17)$$

Furthermore, the function  $\phi$  depends smoothly on the variable  $\sqrt{\epsilon}z$ , and the following estimates hold true: for any integer  $l$  there exists a positive constant  $C_l$  such that

$$\|D_z^l \phi\|_{\epsilon, r-2, \sigma} \leq C_l \left( \sum_{k \leq l} \|D_z^k h\|_{\epsilon, r, \sigma} \right). \quad (4.18)$$

We postpone the proof of Proposition 4.2 to Section 7. We devote the rest of the section to the Proof of Proposition 4.1.

*Proof of Proposition 4.1.* Define

$$\hat{\mathcal{A}} = \Delta_{\mathbb{R}^N} v + \sum_{\ell=0}^2 \mathcal{A}_\ell v + B(v) \quad (4.19)$$

referring to Lemma 3.1, and

$$\Xi_\epsilon(u) = -\hat{\mathcal{A}}u + \epsilon \mu_\epsilon^2 h u - \mu_\epsilon^{\mp \frac{N-2}{2}} u^{p \pm \epsilon} \quad (4.20)$$

We start with  $I = 1$  and the construction of  $w_{1,\epsilon}$  and  $\mu_0$ . A direct computation gives

$$\begin{aligned} \Xi_\epsilon(v_1) &= -\hat{\mathcal{A}}(w_0 + w_{1,\epsilon}) + \epsilon \mu_0^2 h w_0 + \epsilon \mu_0^2 h w_{1,\epsilon} - \mu_0^{\mp \frac{N-2}{2}} (w_0 + w_{1,\epsilon})^{p+\epsilon} \\ &= -\hat{\mathcal{A}}(w_0 + w_{1,\epsilon}) + \epsilon \mu_0^2 h w_0 + \epsilon \mu_0^2 h w_{1,\epsilon} \\ &\quad - \mu_0^{\mp \frac{N-2}{2}} w_0^{p \pm \epsilon} - (p \pm \epsilon) \mu_0^{\mp \frac{N-2}{2}} w_0^{p-1 \pm \epsilon} w_{1,\epsilon} \\ &\quad - \underbrace{\mu_0^{\mp \frac{N-2}{2}} \left[ (w_0 + w_{1,\epsilon})^{p+\epsilon} - w_0^{p \pm \epsilon} - (p \pm \epsilon) w_0^{p-1 \pm \epsilon} w_{1,\epsilon} \right]}_{Q_\epsilon(w_1)} \\ &= -\Delta w_{1,\epsilon} - p w_0^{p-1} w_{1,\epsilon} - \sum_{\ell=0}^2 \mathcal{A}_\ell w_{1,\epsilon} - B(w_{1,\epsilon}) - \sum_{\ell=0}^2 \mathcal{A}_\ell w_0 - B(w_0) \\ &\quad - \underbrace{\left\{ \mu_0^{\mp \frac{N-2}{2}} w_0^{p \pm \epsilon} - w_0^p \right\}}_{I_1} + \epsilon \mu_0^2 h w_0 \\ &\quad - \underbrace{\left\{ (p \pm \epsilon) \mu_0^{\mp \frac{N-2}{2}} w_0^{p-1 \pm \epsilon} w_{1,\epsilon} - p w_0^{p-1} w_{1,\epsilon} \right\}}_{I_2} + \epsilon \mu_0^2 h w_{1,\epsilon} + Q_\epsilon(w_1) \\ &= -\Delta w_{1,\epsilon} - p w_0^{p-1} w_{1,\epsilon} + \epsilon \mu_0^2 h w_{1,\epsilon} + \epsilon \mu_0^2 h w_0 - \underbrace{\mu_0^{\pm \frac{(N-2)^2}{8}} \sum_{\ell=0}^2 \mathcal{A}_\ell w_0}_{I_0} \\ &\quad - \underbrace{\left\{ \mu_0^{\mp \frac{N-2}{2}} w_0^{p \pm \epsilon} - w_0^p \right\}}_{I_1} - \underbrace{\left\{ (p \pm \epsilon) \mu_0^{\mp \frac{N-2}{2}} w_0^{p-1 \pm \epsilon} w_{1,\epsilon} - p w_0^{p-1} w_{1,\epsilon} \right\}}_{I_2} \\ &\quad - B(w_0) - \sum_{\ell=0}^2 \mathcal{A}_\ell w_{1,\epsilon} - B(w_{1,\epsilon}) + Q_\epsilon(w_1). \end{aligned}$$

We next analyze each one of the above terms. Using the expression of the operators  $\mathcal{A}_\ell$ ,  $\ell = 0, \dots, 2$ , given by Lemma 3.1, we get

$$\begin{aligned} I_0 &= \epsilon \left\{ -\mu_0 \Delta_K(\mu_0) Z_0 + |\nabla_K \mu_0|^2 \mathcal{T}_1(w_0) - \mu_0^2 (\mathcal{T}_2(w_0) - \mathcal{T}_3(w_0)) \right\} \\ &\quad + O(\epsilon^2) b(\xi) \end{aligned}$$

where  $b(\xi)$  is a smooth function such that  $\|(1 + |\xi|^{N-2})b(\xi)\|_\infty \leq C$ , for some constant  $C$  independent of  $\epsilon$ . Furthermore, we recall that  $Z_0 = \gamma w_0 + D_\xi w_0[\xi]$ . Also we denoted

$$\mathcal{T}_1(w_0) = D_\xi w_0[\xi]^2 + 2(1 + \gamma)D_\xi w_0[\xi] + \gamma(1 + \gamma)w_0, \quad (4.21)$$

$$\mathcal{T}_2(w_0) = \frac{1}{3} \sum_{i,j} \left[ \sum_{m,l} R_{mijl} \xi_m \xi_l \right] \partial_{ij}^2 w_0, \quad (4.22)$$

$$\mathcal{T}_3(w_0) = \sum_j \left[ \sum_s \frac{2}{3} R_{ssj} + \sum_{m,a,b} (\tilde{g}^{ab} R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \right] \xi_m \partial_j w_0. \quad (4.23)$$

On the other hand, a direct computation shows that

$$I_1 = w_0^p \left[ \mu_0^{\mp \frac{N-2}{2}\epsilon} w_0^{\pm\epsilon} - 1 \right] = \pm \epsilon w_0^p \ln w_0 (1 + O(\epsilon^2)).$$

Thus we can write

$$\Xi_\epsilon(v_1) = -\Delta_{\mathbb{R}^N} w_1 - p w_0^{p-1} w_1 + \epsilon \mu_0^2 h w_1 + \epsilon H_1(z, \xi) + \epsilon L_\epsilon(w_1) + Q_\epsilon(w_1),$$

where

$$\begin{aligned} H_1(z, \xi) &= \mu_0 \Delta_K(\mu_0) Z_0 - |\nabla_K \mu_0|^2 \mathcal{T}_1(w_0) + \mu_0^2 (\mathcal{T}_2(w_0) - \mathcal{T}_3(w_0)) \\ &\quad + \mu_0^2 h w_0 \mp w_0^p \ln(w_0) + \mathcal{E}_{1,\epsilon}, \end{aligned}$$

with  $\mathcal{E}_{1,\epsilon}$  is a sum of functions of the form

$$\epsilon \mu_0 (\epsilon \mu_0 + \epsilon \partial_a \mu_0 + \epsilon \partial_a^2 \mu_0) a(z) b(\xi)$$

and  $a(\epsilon z)$  is a smooth function uniformly bounded, together with its derivatives, as  $\epsilon \rightarrow 0$ , while the function  $b$  is such that

$$\sup_\xi (1 + |\xi|^{N-2}) |b(\xi)| < \infty.$$

The term  $\epsilon L_\epsilon(w_1)$  is linear in  $w_1$ , infact it is explicetly given by

$$\epsilon L_\epsilon(w_1) = I_2 - \sum_{\ell=0}^2 \mathcal{A}_\ell w_{1,\epsilon} - B(w_{1,\epsilon}) \quad (4.24)$$

The term  $Q_\epsilon(w_{1,\epsilon})$  is quadratic in  $w_{1,\epsilon}$ , in fact it is explicetly given by

$$\mu_0^{\mp \frac{N-2}{2}\epsilon} \left[ (w_0 + w_{1,\epsilon})^{p \pm \epsilon} - w_0^{p \pm \epsilon} - p w_0^{p-1 \pm \epsilon} w_{1,\epsilon} \right]. \quad (4.25)$$

We ask the function  $w_{1,\epsilon}$  to satisfy the following equation

$$-\Delta_{\mathbb{R}^N} w_{1,\epsilon} - p w_0^{p-1} w_{1,\epsilon} + \epsilon \mu_0^2 h w_{1,\epsilon} = -\epsilon H_1(z, \xi), \quad \text{in } \mathcal{D}, \quad \phi = 0 \quad \text{on } \partial \hat{\mathcal{D}}. \quad (4.26)$$

Using Proposition 4.2, we see that equation (4.26) is solvable if the right-hand side satisfies the orthogonality conditions in (4.16). These conditions, for  $j = 1, \dots, N$  are clearly satisfied since both  $\xi_j \partial_j w_0$  and  $\partial_{ij}^2 w_0$  are even functions in  $\xi$ , while the  $Z_i$ 's are odd functions in  $\xi$  for every  $i$ . It remains to compute the  $L^2$  product of the right-hand side against  $Z_0$ . Imposing this  $L^2$  product equal to zero will define the function  $\mu_0$ .

We define  $\mu_0$  to satisfy, at main order,

$$\int_{\hat{\mathcal{D}}} H_1(z, \xi) Z_0(\xi) d\xi = 0 \quad \forall z \in K_\epsilon. \quad (4.27)$$



Let us be more precise. We have

$$\begin{aligned} \int_{\hat{\mathcal{D}}} H_1(z, \xi) Z_0(\xi) d\xi &= \mu_0(y) \Delta_K(\mu_0) \int_{\mathbb{R}^N} Z_0^2(\xi) d\xi \\ &- |\nabla_K \mu_0|^2 \int_{\mathbb{R}^N} Z_0(\xi) \mathcal{T}_1(w_0) d\xi + \mu_0^2(y) \int_{\mathbb{R}^N} (\mathcal{T}_2(w_0) - \mathcal{T}_3(w_0)) Z_0(\xi) d\xi \\ &+ \mu_0^2(y) h(y) \int_{\mathbb{R}^N} w_0(\xi) Z_0(\xi) d\xi \mp \int_{\mathbb{R}^N} w_0^p \ln(w_0) Z_0 d\xi + O\left(\left(\frac{\epsilon}{\eta^2}\right)^{\frac{N-4}{2}}\right). \end{aligned}$$

Define

$$c_{1,N} := \int_{\mathbb{R}^N} Z_0^2(\xi) d\xi = \alpha_N^2 \frac{(N-2)^2(N+2)}{2N(N-4)} \omega_N I_N^{N/2} > 0. \quad (4.28)$$

A direct computation gives that

$$\int_{\mathbb{R}^N} Z_0(\xi) \mathcal{T}_1(w_0) d\xi = 0.$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}^N} Z_0(\xi) \mathcal{T}_2(w_0) d\xi &= \frac{1}{3} \sum_{i,j} \sum_{m,l} R_{mijl} \int_{\mathbb{R}^N} \xi_m \xi_l \partial_{ij}^2 w_0 Z_0 d\xi \\ &= \frac{1}{3} \sum_{i,j} R_{jii} \int_{\mathbb{R}^N} \xi_j \partial_j w_0 Z_0 d\xi, \end{aligned}$$

because  $R_{mijl}$  is antisymmetric (i.e.  $R_{mijl} = -R_{imjl}$ ) and

$$\int_{\mathbb{R}^N} \xi_m \xi_l \partial_{ij}^2 w_0 Z_0 d\xi = \alpha_N (N-2) \int_{\mathbb{R}^N} \xi_m \xi_l \left( -\frac{\delta_{ij}}{(1+|\xi|^2)^{\frac{N}{2}}} + \frac{N \xi_i \xi_j}{(1+|\xi|^2)^{\frac{N+2}{2}}} \right) Z_0 d\xi$$

and  $\int_{\mathbb{R}^N} \frac{\xi_m \xi_l \xi_i \xi_j}{(1+|\xi|^2)^{\frac{N+2}{2}}} Z_0 d\xi$  is symmetric. On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^N} Z_0(\xi) \mathcal{T}_3(w_0) d\xi &= \sum_j \left[ \sum_s \frac{2}{3} R_{mssj} + \sum_{m,a,b} (\tilde{g}^{ab} R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \right] \int_{\mathbb{R}^N} \xi_m \partial_j w_0 Z_0 d\xi \\ &= \sum_j \left[ \sum_s \frac{2}{3} R_{jssj} + \sum_{j,a,b} (\tilde{g}^{ab} R_{jabj} - \Gamma_{aj}^b \Gamma_{bj}^a) \right] c_{2,N} \end{aligned} \quad (4.29)$$

where

$$c_{2,N} := \int_{\mathbb{R}^N} \xi_j \partial_j w_0 Z_0 d\xi = \alpha_N^2 \frac{3(N-2)^2}{2N(N-4)} \omega_N I_N^{N/2} > 0. \quad (4.30)$$

From the above computations we deduce

$$\int_{\mathbb{R}^N} Z_0(\xi) (\mathcal{T}_2(w_0) - \mathcal{T}_3(w_0)) d\xi = c_{2,N} \left[ \sum_{i,j} \frac{1}{3} R_{ijij} + \sum_j \sum_{a,b} (\tilde{g}^{ab} R_{jabj} + \Gamma_{aj}^b \Gamma_{bj}^a) \right],$$

We also have

$$\begin{aligned}
\int_{\hat{\mathcal{D}}} \ln(w_0) w_0^p Z_0 d\xi &= \int_{\mathbb{R}^N} \ln(w_0) w_0^p Z_0 d\xi - \int_{\mathbb{R}^N \setminus \hat{\mathcal{D}}} \ln(w_0) w_0^p Z_0 d\xi \\
&= \int_{\mathbb{R}^N} \ln(w_0) w_0^p Z_0 d\xi + O\left(\left(\frac{\epsilon}{\eta}\right)^{\frac{N-2}{2}}\right) \\
&:= c_{4,N} + O\left(\left(\frac{\epsilon}{\eta}\right)^{\frac{N-2}{2}}\right),
\end{aligned}$$

where

$$c_{4,N} := \frac{N}{(p+1)^2} \int_{\mathbb{R}^N} w_0^{p+1}(\xi) d\xi = \alpha_N^{p+1} \frac{(N-2)^3}{4N^2} \omega_N I_N^{N/2} > 0. \quad (4.31)$$

Finally, set

$$c_{3,N} := \int_{\mathbb{R}^N} w_0(\xi) Z_0(\xi) d\xi = -\alpha_N^2 \frac{2(N-1)(N-2)}{N(N-4)} \omega_N I_N^{N/2} < 0. \quad (4.32)$$

We define  $\mu_0$  to satisfy

$$-\Delta_K \mu_0 + \frac{c_{3,N}}{c_{1,N}} \mathcal{H}(y) \mu_0 \pm \frac{c_{4,N}}{c_{1,N}} \frac{1}{\mu_0} = 0, \quad \text{in } K. \quad (4.33)$$

where  $\mathcal{H}$  is the function defined in Fermi coordinates by

$$\mathcal{H}(y) = h(y) - \hat{\Omega}(y) \quad (4.34)$$

where

$$\hat{\Omega}(y) = -\frac{3(N-2)}{4(N-1)} \left[ \sum_{i,j=1}^N \frac{1}{3} R_{jijj}(y) + \sum_{i=1}^N \sum_{a,b=1}^k (\tilde{g}^{ab} R_{iaib}(y) + \Gamma_{ai}^b(y) \Gamma_{bi}^a(y)) \right].$$

The existence of  $\mu_0$  is guaranteed by our assumption. With this choice for  $\mu_0$ , the integral of the right hand side in (4.26) against  $Z_0$  vanishes on  $K$  and this implies the existence of  $w_{1,\epsilon}$ , thanks to Proposition 4.2. Moreover, it is straightforward to check that

$$\|H_1(z, \xi)\|_{\epsilon, N-2, \sigma} \leq C$$

for some  $\sigma \in (0, 1)$ . Proposition 4.2 thus gives that

$$\|D_\xi^2 w_{1,\epsilon}\|_{\epsilon, N-2, \sigma} + \|D_\xi w_{1,\epsilon}\|_{\epsilon, N-3, \sigma} + \|w_{1,\epsilon}\|_{\epsilon, N-4, \sigma} \leq C\epsilon \quad (4.35)$$

and that there exists a positive constant  $\beta$  (depending only on  $K$  and  $N$ ) such that for any integer  $\ell$  there holds

$$\|\nabla_z^{(\ell)} w_{1,\epsilon}(z, \cdot)\|_{\epsilon, N-4, \sigma} \leq \beta C_l \epsilon \quad \sqrt{\epsilon} z \in K \quad (4.36)$$

where  $C_l$  depends only on  $l$ ,  $p$  and  $K$ .

With this choice of  $\mu_0$  and  $w_{1,\epsilon}$  we get that

$$\| -\mathcal{A}_{\mu_\epsilon, \Phi_\epsilon} v_{1,\epsilon} + \epsilon \mu_0^2 h v_{2,\epsilon} - \mu_0^{\mp \epsilon \frac{N-2}{2}} v_{1,\epsilon}^{p \pm \epsilon} \|_{\epsilon, N-2, \sigma} \leq C\epsilon. \quad (4.37)$$

To prove this, we first observe that

$$\|\mu_0^2 \Delta_{K_\epsilon} w_{1,\epsilon}\|_{\epsilon, N-2, \sigma} \leq C\epsilon,$$

as consequence of (4.36). Then we claim that  $\|\epsilon L_\epsilon(w_{1,\epsilon}) + Q_\epsilon(w_{1,\epsilon})\|_{\epsilon, N-2} \leq C\epsilon$ , see (4.24) and (4.25). Indeed, first observe that

$$I_2 = (p \pm \epsilon) \mu_0^{\mp \frac{N-2}{2} \epsilon} w_0^{p-1 \pm \epsilon} w_{1,\epsilon} - p w_0^{p-1} w_{1,\epsilon} = O(\epsilon) w_0^{p-1} w_{1,\epsilon}.$$

and then easily we get that  $|I_2| \leq C \frac{\epsilon}{(1+|\xi|)^{N-2}}$ . Analogous consideration gives that  $\|Q_\epsilon(w_{1,\epsilon})\|_{\epsilon, N-2} \leq C\epsilon$ . Furthermore, using the fact that  $|\xi| \leq \eta\epsilon^{-\frac{1}{2}}$ , we have that

$$\left| \sum_{\ell=0}^2 \mathcal{A}_\ell(w_{1,\epsilon}) + B(w_{1,\epsilon}) \right| \leq C \frac{\epsilon}{(1+|\xi|)^{N-2}}.$$

Estimate (4.37) follows from the regularity of the function  $w_{1,\epsilon}$ .

Let  $I = 2$ . Then  $\mu_{2,\epsilon} = \mu_0 + \epsilon\mu_1$ ,  $\Phi_\epsilon = \Phi_{1,\epsilon}$ , and  $v_2 = w_0 + w_{1,\epsilon} + w_{2,\epsilon}$ , where  $\mu_0$  and  $w_{1,\epsilon}$  have already been constructed in the previous step. Computing  $\Xi_\epsilon(v_2)$ ,

$$\begin{aligned} \Xi_\epsilon(v_2) &= -\Delta_{\mathbb{R}^N} w_{2,\epsilon} - p w_0^{p-1} w_{2,\epsilon} + \epsilon \mu_{2,\epsilon}^2 h w_{2,\epsilon} + H_2(z, \xi) \\ &+ \epsilon^2 L_\epsilon(w_{2,\epsilon}) + Q_\epsilon(w_{2,\epsilon}), \end{aligned} \quad (4.38)$$

where

$$\begin{aligned} H_2(z, \xi) &= \epsilon^2 \left\{ \mu_0 \Delta_K(\mu_1) Z_0 + \mu_1 \Delta_K(\mu_0) Z_0 - \nabla_K(\mu_0) \nabla_K(\mu_1) \mathcal{T}_1(w_0) \right. \\ &\quad + 2\mu_0 \mu_1 (\mathcal{T}_2(w_0) - \mathcal{T}_3(w_0)) + 2\mu_0 \mu_1 h w_0 \\ &\quad \left. \pm \frac{(N-2)^2}{16} w_0^p (\ln(w_0) + 1) \right\} \\ &+ \epsilon^{\frac{3}{2}} \mu_0 \left\{ -\Delta_K \Phi_{1,\epsilon} D_\xi w_0 + \frac{1}{3} \sum_{i,j} \sum_{m,l} R_{mijl} (\xi_m \Phi_{1,\epsilon}^l + \xi_l \Phi_{1,\epsilon}^m) \partial_{ij}^2 w_0, \right. \\ &\quad \left. + \sum_j \left[ \sum_s \frac{2}{3} R_{mssj} + \sum_{m,a,b} (\tilde{g}^{ab} R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \right] \Phi_{1,\epsilon}^m \partial_j w_0 \right\} \\ &+ \epsilon^{\frac{3}{2}} \mathcal{E}_{2,\epsilon}(y, \xi, w_0, w_{1,\epsilon}, \mu_0), \end{aligned}$$

and  $\mathcal{E}_{2,\epsilon}$  is a sum of functions of the form

$$(\mu_0 + \partial_a \mu_0 + \partial_a^2 \mu_0) a(\sqrt{\epsilon} z) b(\xi)$$

and  $a(\sqrt{\epsilon} z)$  is a smooth function uniformly bounded, together with its derivatives, as  $\epsilon \rightarrow 0$ , while the function  $b$  is such that

$$\sup_{\xi} (1 + |\xi|^{N-2}) |b(\xi)| < \infty.$$

Furthermore, we have

$$\begin{aligned} \epsilon^2 L_\epsilon(w_2) &= -\mu_0^{\mp \frac{N-2}{2} \epsilon} \left[ (w_0 + w_{1,\epsilon} + w_{2,\epsilon})^{p+\epsilon} \right. \\ &\quad \left. - (w_0 + w_{1,\epsilon})^{p \pm \epsilon} - (p \pm \epsilon)(w_0 + w_{1,\epsilon})^{p-1 \pm \epsilon} w_{2,\epsilon} \right] \\ &- \sum_{\ell=0}^2 \mathcal{A}_\ell w_{2,\epsilon} - B(w_{2,\epsilon}) \end{aligned}$$

and the term  $Q_\epsilon(w_{2,\epsilon})$  in (4.38) is a sum of quadratic terms in  $w_{2,\epsilon}$  like

$$\begin{aligned} (\mu_0 + \epsilon\mu_1)^{\mp \frac{N-2}{2} \epsilon} \left[ (w_0 + w_{1,\epsilon} + w_{2,\epsilon})^{p \pm \epsilon} - (w_0 + w_{1,\epsilon})^{p \pm \epsilon} \right. \\ \left. - (p \pm \epsilon)(w_0 + w_{1,\epsilon})^{p-1 \pm \epsilon} w_{2,\epsilon} \right]. \end{aligned}$$

We will choose  $w_{2,\epsilon}$  to satisfy the following equation

$$-\Delta_{\mathbb{R}^N} w_{2,\epsilon} - p w_0^{p-1} w_{2,\epsilon} + \epsilon \mu_{2,\epsilon}^2 h w_{2,\epsilon} = -H_2(z, \xi), \text{ in } \mathcal{D}, \quad w_{2,\epsilon} = 0 \quad \text{on } \partial \hat{\mathcal{D}}. \quad (4.39)$$

Thanks to Proposition 4.2, we see that equation (4.39) is solvable if the right-hand side is  $L^2$ -orthogonal to the functions  $Z_j$ , for  $j = 0, \dots, N$ . These orthogonality conditions will define the parameters  $\mu_1$  and the normal section  $\Phi_{1,\epsilon}$ .

Projection onto  $Z_0$  and choice of  $\mu_1$ : the function  $\mu_1$  is asked to satisfy, at main order,

$$\int_{\hat{D}} H_{2,\epsilon} Z_0 d\xi = 0.$$

Computations similar to the ones already performed to define  $\mu_0$  give that  $\mu_1$  satisfies

$$-\mu_0(y) \Delta_K \mu_1(y) - \mu_1 \Delta_K \mu_0(y) + 2\mu_0(y) \mu_1(y) g(y) \pm \frac{(N-2)^2}{16} \frac{c_{4,N}}{c_{1,N}} = 0 \quad (4.40)$$

in  $K$ , where  $g(y)$  is given by (4.34), and  $c_{i,N}$  ( $i = 1, 2, 3, 4$ ) are defined in (4.28), (4.30), (4.32) and (4.31). According to our choice of  $\mu_0$  satisfies (4.33), then (4.40) is equivalent to the following equation

$$-\Delta_K \mu_1(y) + g(y) \mu_1(y) \mp \frac{c_{4,N}}{c_{1,N}} \frac{1}{\mu_0^2(y)} \mu_1(y) = \mp \frac{(N-2)^2}{16} \frac{c_{4,N}}{c_{1,N}} \frac{1}{\mu_0(y)}, \quad (4.41)$$

in  $K$ . The existence of  $\mu_1$  is guaranteed by the nondegeneracy of  $\mu_0$ .

Projection onto  $Z_s$  and choice of  $\Phi_{1,\epsilon}$ . Multiplying  $H_{2,\epsilon}$  with  $Z_s = \partial_s w_0$ , integrating over  $\hat{D}$  and using the fact  $w_0$  is even in the variable  $\xi$ , one obtains

$$\begin{aligned} \int_{\hat{D}} H_{2,\epsilon} \partial_s w_0 d\xi &= -\epsilon^{\frac{3}{2}} \mu_0 \sum_j \Delta_K \Phi_{1,\epsilon}^j \int_{\hat{D}} \partial_j w_0 \partial_s w_0 d\xi + \epsilon^{\frac{3}{2}} \int_{\hat{D}} \mathcal{E}_{2,\epsilon} \partial_s w_0 d\xi \\ &\quad + \epsilon^{\frac{3}{2}} \frac{\mu_0}{3} \sum_{i,j} \sum_{m,l} R_{mijl} \int_{\hat{D}} (\xi_m \Phi_{1,\epsilon}^l + \xi_l \Phi_{1,\epsilon}^m) \partial_{ij}^2 w_0 \partial_s w_0 d\xi \\ &\quad + \epsilon^{\frac{3}{2}} \frac{2\mu_0}{3} \sum_{j,m} \sum_l R_{mllj} \Phi_{1,\epsilon}^m \int_{\hat{D}} \partial_j w_0 \partial_s w_0 \\ &\quad + \epsilon^{\frac{3}{2}} \sum_{j,m} \sum_{a,b} (\tilde{g}^{ab} R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \Phi_{1,\epsilon}^m \int_{\hat{D}} \partial_j w_0 \partial_s w_0. \end{aligned} \quad (4.42)$$

First of all, observe that by oddness in  $\xi$  we have that

$$\int_{\hat{D}} \partial_j w_0 \partial_s w_0 = \delta_{js} C_0 + O(\epsilon^{\frac{N-2}{2}}), \quad C_0 := \int_{\mathbb{R}^N} |\partial_l w_0|^2 d\xi.$$

Thus

$$-\epsilon^{\frac{3}{2}} \mu_0 \sum_j \Delta_K \Phi_{1,\epsilon}^j \int_{\hat{D}} \partial_j w_0 \partial_s w_0 d\xi = -\epsilon^{\frac{3}{2}} \mu_0 C_0 \Delta_K \Phi_{1,\epsilon}^s + O(\epsilon^{\frac{N-2}{2}})$$

and

$$\epsilon^{\frac{3}{2}} \frac{2\mu_0}{3} \sum_{j,m} \sum_l R_{mllj} \Phi_{1,\epsilon}^m \int_{\hat{D}} \partial_j w_0 \partial_s w_0 = \epsilon^{\frac{3}{2}} \frac{2\mu_0}{3} C_0 \sum_m \sum_l R_{mlls} \Phi_{1,\epsilon}^m.$$

On the other hand the integral  $\int_{\hat{D}} \xi_m \partial_{ij}^2 w_0 \partial_s w_0$  is non-zero only if, either  $i = j$  and  $m = s$ , or  $i = s$  and  $j = m$ , or  $i = m$  and  $j = s$ . In the latter case we have  $R_{mij s} = 0$  (by the antisymmetry

of the curvature tensor in the first two indices). Therefore, the first term of the second line in (4.42) becomes simply

$$\begin{aligned} & 2\epsilon^{\frac{3}{2}} \frac{\mu_0}{3} \sum_m \sum_l R_{mlls} \left( - \int \xi_l \partial_{ls}^2 w_0 \partial_s w_0 + \int \xi_s \partial_{ll}^2 w_0 \partial_s w_0 \right) \Phi_{1,\epsilon}^m + O(\epsilon^{\frac{N-2}{2}}) \\ &= -2\epsilon^{\frac{3}{2}} \frac{\mu_0}{3} \sum_m \sum_l R_{mlls} \left( - \int \xi_l \partial_{ls}^2 w_0 \partial_s w_0 + \int \xi_s \partial_{ls}^2 w_0 \partial_l w_0 \right) \Phi_{1,\epsilon}^m + O(\epsilon^{\frac{N-2}{2}}) \\ &= -2\epsilon^{\frac{3}{2}} \frac{\mu_0}{3} C_0 \sum_m \sum_l R_{mlls} \Phi_{1,\epsilon}^m + O(\epsilon^{\frac{N-2}{2}}) \end{aligned}$$

where the last identity is consequence of the following fact

$$\int \xi_l \partial_{sl}^2 w_0 \partial_s w_0 = -\frac{C_0}{2},$$

which can be proved with a straightforward computation.

Hence formula (4.42) becomes simply

$$\begin{aligned} \int_{\mathcal{D}} H_{2,\epsilon} \partial_s w_0 d\xi &= \epsilon^{\frac{3}{2}} \mu_0 C_0 \left[ -\Delta_K \Phi_{1,\epsilon}^s + \sum_m \sum_{a,b} \left( \tilde{g}^{ab} R_{mabs} - \Gamma_a^b(E_m) \Gamma_b^a(E_s) \right) \Phi_{1,\epsilon}^m \right] \\ &\quad + O(\epsilon^{N-\frac{1}{2}}) \Phi_{1,\epsilon}^m + \epsilon^{\frac{3}{2}} \int_{\mathcal{D}} \mathcal{E}_{2,\epsilon} \partial_s w_0. \end{aligned}$$

We thus obtain that  $\int H_{2,\epsilon}(z, \xi, w_0, \dots, w_{1,\epsilon}) Z_l = 0$  at main order if  $\Phi_{1,\epsilon}$  satisfies an equation of the form

$$\Delta_K \Phi_{1,\epsilon}^s - \sum_m \sum_{a,b} \left( \tilde{g}^{ab} R_{mabs} - \Gamma_a^b(E_m) \Gamma_b^a(E_s) \right) \Phi_{1,\epsilon}^m = G_{2,\epsilon}(\sqrt{\epsilon}z), \quad (4.43)$$

for some expression  $G_{2,\epsilon}$  smooth on its argument. Observe that the operator acting on  $\Phi_{1,\epsilon}$  in the left hand side is nothing but the Jacobi operator, which is invertible by the non-degeneracy condition on  $K$ . This implies the solvability of the above equation in  $\Phi_{1,\epsilon}$ . Furthermore, equation (4.43) defines  $\Phi_{1,\epsilon}$  as a smooth function on  $K$ , of order  $\epsilon$ , more precisely we have

$$\|\Phi_{1,\epsilon}\|_{\infty} + \|\partial_a \Phi_{1,\epsilon}\|_{\infty} + \|\partial_a^2 \Phi_{1,\epsilon}\|_{\infty} \leq C. \quad (4.44)$$

By our choice of  $\mu_1$  and  $\Phi_{1,\epsilon}$  we have solvability of equation (4.39) in  $w_{2,\epsilon}$ . Moreover, it is straightforward to check that

$$|H_{2,\epsilon}(z, \xi)| \leq C \max \left\{ \epsilon^2, \epsilon^{\frac{3}{2}} \right\} \frac{1}{(1 + |\xi|)^{N-2}}.$$

Furthermore, for a given  $\sigma \in (0, 1)$  we have

$$\|H_{2,\epsilon}\|_{\epsilon, N-2, \sigma} \leq C \epsilon^{\frac{3}{2}}.$$

Proposition 4.2 thus gives then that

$$\|D_{\xi}^2 w_{2,\epsilon}\|_{\epsilon, N-2, \sigma} + \|D_{\xi} w_{2,\epsilon}\|_{\epsilon, N-3, \sigma} + \|w_{2,\epsilon}\|_{\epsilon, N-4, \sigma} \leq C \epsilon^{\frac{3}{2}} \quad (4.45)$$

and that there exists a positive constant  $\beta$  (depending only on  $\Omega, K$  and  $n$ ) such that for any integer  $\ell$  there holds

$$\|\nabla_z^{(\ell)} w_{2,\epsilon}(z, \cdot)\|_{\epsilon, N-4, \sigma} \leq \beta C_{\ell} \epsilon^{\frac{3}{2}},$$

where  $C_{\ell}$  depends only on  $\ell, p, K$ .

Arguing as in the previous step for  $I = 1$ , we see that with this choice of  $\mu_{1,\epsilon}$ ,  $\Phi_{1,\epsilon}$  and  $w_{2,\epsilon}$  we get that

$$\| -\mathcal{A}_{\mu_{\epsilon}, \Phi_{\epsilon}} v_{2,\epsilon} + \epsilon \mu_{\epsilon}^2 h v_{2,\epsilon} - \mu_{\epsilon}^{\mp \epsilon^{\frac{N-2}{2}}} v_{2,\epsilon}^{p \pm \epsilon} \|_{\epsilon, N-2, \sigma} \leq C \epsilon^{\frac{3}{2}}.$$

Expansion at an arbitrary order. We take now an arbitrary integer  $I$ . Let

$$\mu_{I+1,\epsilon} := \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 \cdots + \epsilon^{I-1}\mu_{I-1} + \epsilon^I\mu_I, \quad (4.46)$$

$$\Phi_\epsilon = \Phi_{1,\epsilon} + \cdots + \Phi_{I-1,\epsilon} + \Phi_{I,\epsilon} \quad (4.47)$$

and

$$v_{I+1,\epsilon} = w_0(\xi) + w_{1,\epsilon}(z, \xi) + \cdots + w_{I,\epsilon}(z, \xi) + w_{I+1,\epsilon}(z, \xi) \quad (4.48)$$

where  $\mu_0, \mu_1, \dots, \mu_{I-1,\epsilon}, \Phi_{1,\epsilon}, \dots, \Phi_{I,\epsilon}$  and  $w_{1,\epsilon}, \dots, w_{I,\epsilon}$  have already been constructed following an iterative scheme, as described in the previous steps of the construction.

In particular one has, for any  $i = 1, \dots, I-1$

$$\|\mu_i\|_\infty + \|\partial_a \mu_i\|_\infty + \|\partial_a^2 \mu_i\|_\infty \leq C \quad (4.49)$$

$$\|\Phi_{i,\epsilon}\|_\infty + \|\partial_a \Phi_{i,\epsilon}\|_\infty + \|\partial_a^2 \Phi_{i,\epsilon}\|_\infty \leq C\epsilon^{i-1} \quad (4.50)$$

We have

$$\|D_\xi^2 w_{i,\epsilon}\|_{\epsilon, N-2, \sigma} + \|D_\xi w_{i,\epsilon}\|_{\epsilon, N-3, \sigma} + \|w_{i,\epsilon}\|_{\epsilon, N-4, \sigma} \leq C\epsilon^{i-\frac{1}{2}}. \quad (4.51)$$

and, for any integer  $\ell$ ,  $\sqrt{\epsilon}z \in K$ ,

$$\|\nabla_z^{(\ell)} w_{i,\epsilon}(z, \cdot)\|_{\epsilon, N-2, \sigma} \leq \beta C_\ell \epsilon^{i-\frac{1}{2}}. \quad (4.52)$$

The new triplet  $(\mu_I, \Phi_{I,\epsilon}, w_{I+1,\epsilon})$  will be found reasoning as in the construction of  $(\mu_1, \Phi_1, w_{2,\epsilon})$ . Computing  $\Xi_\epsilon(v_{I+1,\epsilon})$  we get

$$-\hat{\mathcal{A}}_{\mu_\epsilon, \Phi_\epsilon} v_{I+1,\epsilon} + \epsilon\mu_{I+1,\epsilon}^2 h v_{I+1,\epsilon} - \mu_{I+1,\epsilon}^{\mp \epsilon \frac{N-2}{2}} v_{I+1,\epsilon}^{p \pm \epsilon} \quad (4.53)$$

$$= -\Delta_{\mathbb{R}^N} w_{I+1,\epsilon} - p w_0^{p-1} w_{I+1,\epsilon} + \epsilon\mu_{I+1,\epsilon}^2 h w_{I+1,\epsilon} + H_{I+1}(z, \xi) + Q_\epsilon(w_{I+1,\epsilon})$$

where the function  $H_{I+1,\epsilon}$  is given by

$$\begin{aligned} H_{I+1,\epsilon} &= \epsilon^{I+1} \left\{ \mu_0 \Delta_K(\mu_I) Z_0 + \mu_I \Delta_K(\mu_0) Z_0 - \nabla_K(\mu_0) \nabla_K(\mu_I) \mathcal{T}_1(w_0) \right. \\ &\quad + 2\mu_0 \mu_I (\mathcal{T}_2(w_0) - \mathcal{T}_3(w_0)) + 2\mu_0 \mu_I h w_0 \\ &\quad \left. \pm \frac{(N-2)^2}{16} w_0^p (\ln(w_0) + 1) \right\} \\ &\quad + \epsilon^{I+\frac{1}{2}} \mu_0 \left\{ -\Delta_K \Phi_{I,\epsilon} D_\xi w_0 + \frac{1}{3} \sum_{i,j} \sum_{m,l} R_{mijl} (\xi_m \Phi_{I,\epsilon}^l + \xi_I \Phi_{I,\epsilon}^m) \partial_{ij}^2 w_0, \right. \\ &\quad \left. + \sum_j \left[ \sum_s \frac{2}{3} R_{mssj} + \sum_{m,a,b} (R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \right] \Phi_{I,\epsilon}^m \partial_j w_0 \right\} \\ &\quad + \epsilon^{I+\frac{1}{2}} \mathcal{E}_{I+1,\epsilon}(y, \xi, w_0, w_{1,\epsilon}, \dots, w_{I,\epsilon}, \mu_0, \dots, \mu_{I-1}). \end{aligned} \quad (4.54)$$

In (4.54),  $\mathcal{E}_{I+1,\epsilon}$  is a sum of functions of the form

$$\begin{aligned} &\left[ (\mu_0 + \partial_a \mu_0 + \partial_a^2 \mu_0 + \cdots + \mu_{I-1} + \partial_a \mu_{I-1} + \partial_a^2 \mu_{I-1}) \right. \\ &\quad \left. + (\Phi_1 + \partial_a \Phi_1 + \partial_a^2 \Phi_1 + \cdots + \Phi_{I-2} + \partial_a \Phi_{I-2} + \partial_a^2 \Phi_{I-2}) \right] a(\sqrt{\epsilon}z) b(\xi) \end{aligned}$$

where  $a(\sqrt{\epsilon}z)$  is a smooth function uniformly bounded, together with its derivatives, as  $\epsilon \rightarrow 0$ , while the function  $b$  is such that

$$\sup_\xi (1 + |\xi|^{N-2}) |b(\xi)| < \infty.$$

Finally the term  $Q_\epsilon(w_{I,\epsilon})$  in (4.53) is a sum of quadratic terms in  $w_{I+1,\epsilon}$  like

$$\begin{aligned} & (\mu_0 + \epsilon\mu_1 + \dots + \epsilon^I \mu_I)^{\mp \frac{N-2}{2}\epsilon} \times \\ & [(w_0 + w_{1,\epsilon} + w_{2,\epsilon} + \dots + w_{I+1,\epsilon})^{p \pm \epsilon} \\ & - (w_0 + w_{1,\epsilon} + \dots + w_{I,\epsilon})^{p \pm \epsilon} \\ & - (p \pm \epsilon)(w_0 + w_{1,\epsilon} + \dots + w_{I,\epsilon})^{p-1 \pm \epsilon} w_{I+1,\epsilon}] \end{aligned}$$

and linear terms in  $w_{I,\epsilon}$  multiplied by a term of order  $\epsilon$ , like

$$p \left( (w_0 + w_{1,\epsilon} + \dots + w_{I-1,\epsilon})^{p-1 \pm \epsilon} - w_0^{p-1 \pm \epsilon} \right) w_{I,\epsilon}.$$

Consider the following problem

$$\begin{aligned} -\Delta_{\mathbb{R}^N} w_{I+1,\epsilon} - p w_0^{p-1} w_{I+1,\epsilon} + \epsilon \mu_{I+1,\epsilon}^2 h w_{I+1,\epsilon} &= -H_{I+1,\epsilon}(z, \xi) \quad \text{in } \mathcal{D} \\ w_{I+1,\epsilon} &= 0 \quad \text{on } \partial \hat{\mathcal{D}}. \end{aligned} \quad (4.55)$$

Again by Proposition 4.2, the above problem is solvable in  $w_{I+1,\epsilon}$  if  $H_{I+1,\epsilon}$  is  $L^2$ -orthogonal to  $Z_j$ ,  $j = 0, 1, \dots, N$ . These orthogonality conditions will define the parameters  $\mu_I$  and the normal section  $\Phi_{I,\epsilon}$ .

Projection onto  $Z_0$  and choice of  $\mu_I$ . We define  $\mu_I$  to make, at main order,

$$\int_{\hat{\mathcal{D}}} H_{I+1,\epsilon} Z_0 d\xi = 0.$$

The above relation defines  $\mu_I$  as a smooth function of  $\sqrt{\epsilon}z$  in  $K$ . From estimates (4.51) we get that

$$\|\mu_I\|_\infty + \|\partial_a \mu_I\|_\infty + \|\partial_a^2 \mu_I\|_\infty \leq C \quad (4.56)$$

Projection onto  $Z_s$  and choice of  $\Phi_{I,\epsilon}$ . Multiplying  $H_{I+1,\epsilon}$  with  $\partial_s w_0$ , integrating over  $\hat{\mathcal{D}}$  and arguing as in the construction of  $\Phi_{I,\epsilon}$ , we get

$$\begin{aligned} & \left( C_0 \epsilon^{I+\frac{1}{2}} \mu_0 \right)^{-1} \int_{\hat{\mathcal{D}}} H_{I+1,\epsilon} \partial_s w_0 d\xi = -\Delta_K \Phi_{I,\epsilon}^l \\ & + \sum_m \sum_{a,b} \left( \tilde{g}^{ab} R_{maal} - \Gamma_a^b(E_m) \Gamma_b^a(E_l) + O(\epsilon^{\frac{N-2}{2}}) \right) \Phi_{I,\epsilon}^m + \int_{\hat{\mathcal{D}}} \mathfrak{G}_{I+1,\epsilon} \partial_l w_0. \end{aligned}$$

We then conclude that  $H_{I+1,\epsilon}(z, \xi, w_0, \dots, w_{I,\epsilon})$ , the right-hand side of (4.55), is  $L^2$ -orthogonal to  $Z_l$  ( $l = 1, \dots, N$ ) if and only if  $\Phi_{I,\epsilon}$  satisfies an equation of the form

$$\begin{aligned} & \Delta_K \Phi_{I,\epsilon}^l - \sum_m \sum_{a,b} \left( \tilde{g}^{ab} R_{maal} - \Gamma_a^b(E_m) \Gamma_b^a(E_l) + O(\epsilon^{\frac{N-2}{2}}) \right) \Phi_{I,\epsilon}^m \\ & = \epsilon^{I+\frac{1}{2}} G_{I+1,\epsilon}(\sqrt{\epsilon}z), \end{aligned} \quad (4.57)$$

where  $G_{I+1,\epsilon}$  is a smooth function on  $K$ , uniformly bounded as  $\epsilon \rightarrow 0$ . Using again the non-degeneracy condition on  $K$  we have solvability of the above equation in  $\Phi_{I,\epsilon}$ . Furthermore, taking into account (4.53), we get

$$\|\Phi_{I,\epsilon}\|_\infty + \|\partial_a \Phi_{I,\epsilon}\|_\infty + \|\partial_a^2 \Phi_{I,\epsilon}\|_\infty \leq C \epsilon^{I+\frac{1}{2}}. \quad (4.58)$$

By our choice of  $\mu_{I+1}$  and  $\Phi_{I+1,\epsilon}$  we have solvability of equation (4.55) in  $w_{I+1,\epsilon}$ . Moreover, it is straightforward to check that

$$|H_{I+1,\epsilon}(\epsilon z, \xi)| \leq C \epsilon^{I+\frac{1}{2}} \frac{1}{(1 + |\xi|)^{N-2}}.$$

Furthermore, for a given  $\sigma \in (0, 1)$  we have

$$\|H_{I+1,\epsilon}\|_{\epsilon,N-2,\sigma} \leq C\epsilon^{I+\frac{1}{2}}.$$

Proposition 4.2 gives then that

$$\|D_\xi^2 w_{I+1,\epsilon}\|_{\epsilon,N-2,\sigma} + \|D_\xi w_{I+1,\epsilon}\|_{\epsilon,N-3,\sigma} + \|w_{I+1,\epsilon}\|_{\epsilon,N-4,\sigma} \leq C\epsilon^{I+\frac{1}{2}}, \quad (4.59)$$

and that there exists a positive constant  $\beta$  (depending only on  $K$  and  $N$ ) such that for any integer  $\ell$  there holds, for  $\sqrt{\epsilon}z \in K$ ,

$$\|\nabla^{(\ell)} w_{I+1,\epsilon}(z, \cdot)\|_{\epsilon,N-2,\sigma} \leq \beta C_l \epsilon^{I+\frac{1}{2}}.$$

With this choice of  $\mu_I$ ,  $\Phi_{I,\epsilon}$  and  $w_{I+1,\epsilon}$  we obtain that

$$\| -\mathcal{A}_{\mu_\epsilon, \Phi_\epsilon} v_{I+1,\epsilon} + \epsilon \mu_{I+1,\epsilon}^2 h v_{I+1,\epsilon} - \mu_\epsilon^{\mp \epsilon \frac{N-2}{2}} v_{I+1,\epsilon}^{p \pm \epsilon} \|_{\epsilon,N-2,\sigma} \leq C\epsilon^{I+\frac{1}{2}}.$$

This concludes our construction. □

## 5. A GLOBAL APPROXIMATION

Let us recall that if  $u$  is a solution to problem (1.1), and we define

$$u(x) = (1 + \alpha_\epsilon) \epsilon^{-\frac{N-2}{4}} \tilde{u}(\epsilon^{-\frac{1}{2}} x).$$

Then  $\tilde{u}$  satisfies the following equation

$$-\Delta_{g^\epsilon} \tilde{u} + \epsilon h \tilde{u} = \tilde{u}^{\frac{N+2}{N-2} \pm \epsilon} \quad \text{in } \mathcal{M}_\epsilon; \quad \tilde{u} > 0 \text{ in } \mathcal{M}_\epsilon, \quad (5.1)$$

where  $\Delta_{g^\epsilon}$  denotes the Laplace-Beltrami operator on  $\mathcal{M}_\epsilon$  is given by

$$\Delta_{g^\epsilon} = \frac{1}{\sqrt{\det g^\epsilon}} \partial_A (\sqrt{\det g^\epsilon} (g^\epsilon)^{AB} \partial_B)$$

here indices  $A$  and  $B$  run between 1 and  $n = N + k$ , and  $g^\epsilon$  is the scaled metric on  $\mathcal{M}_\epsilon$  whose coefficient are defined by

$$g_{\alpha,\beta}^\epsilon(z, x) = g_{\alpha,\beta}(\sqrt{\epsilon}z, \sqrt{\epsilon}x)$$

where  $g_{\alpha,\beta}$  are the coefficients of the metric  $g$  on  $\mathcal{M}$ .

Let  $\mu_\epsilon(y)$ ,  $\Phi_\epsilon(y)$  and  $v_{I+1,\epsilon}$  be the functions whose existence and properties have been established in Lemma 4.1. We define locally around  $K_\epsilon := \frac{K}{\sqrt{\epsilon}}$  the function

$$\tilde{U}_\epsilon(z, x) := \mu_\epsilon^{-\frac{N-2}{2}} (\sqrt{\epsilon}z) v_{I+1,\epsilon} \left( z, \frac{x - \sqrt{\epsilon} \Phi_\epsilon(\sqrt{\epsilon}z)}{\mu_\epsilon(\sqrt{\epsilon}z)} \right) \chi_\epsilon(|(x - \sqrt{\epsilon} \Phi_\epsilon(\sqrt{\epsilon}z))|) \quad (5.2)$$

where  $z \in K_\epsilon$ . The function  $\chi_\epsilon$  is a smooth cut-off function with

$$\chi_\epsilon(r) = \begin{cases} 1, & \text{for } r \in [0, 2\epsilon^{-\gamma}] \\ 0, & \text{for } r \in [3\epsilon^{-\gamma}, 4\epsilon^{-\gamma}], \end{cases} \quad (5.3)$$

and

$$|\chi_\epsilon^{(l)}(r)| \leq C_l \epsilon^{l\gamma}, \quad \forall l \geq 1,$$

for some  $\gamma \in (\frac{1}{2}, 1)$  to be fixed later.

We will use the notation

$$\tilde{u} = \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(\tilde{v}) \quad (5.4)$$

if and only if  $\tilde{u}$  and  $\tilde{v}$  satisfy

$$\tilde{u} = \mu_\epsilon^{-\frac{N-2}{2}} (\sqrt{\epsilon}z) \tilde{v} \left( z, \frac{x - \sqrt{\epsilon} \Phi_\epsilon(\sqrt{\epsilon}z)}{\mu_\epsilon(\sqrt{\epsilon}z)} \right).$$



The function  $\tilde{U}_\epsilon$  is well defined in a small neighborhood of  $K_\epsilon$ . We will look for a solution to (5.1) of the form

$$\tilde{u}_\epsilon = \tilde{U}_\epsilon + \phi.$$

Thus  $\phi$  satisfies the following problem

$$-\Delta_{g^\epsilon}\phi + \epsilon h\phi - (p \pm \epsilon)\tilde{U}_\epsilon^{p \pm \epsilon - 1}\phi = S_\epsilon(\tilde{U}_\epsilon) + N_\epsilon(\phi) \quad \text{in } \mathcal{M}_\epsilon, \quad (5.5)$$

where

$$S_\epsilon(\tilde{U}_\epsilon) = \Delta_{g^\epsilon}\tilde{U}_\epsilon - \epsilon h\tilde{U}_\epsilon + \tilde{U}_\epsilon^p \quad (5.6)$$

and

$$N_\epsilon(\phi) = (\tilde{U}_\epsilon + \phi)^{p \pm \epsilon} - \tilde{U}_\epsilon^p - (p \pm \epsilon)\tilde{U}_\epsilon^{p \pm \epsilon - 1}\phi. \quad (5.7)$$

Define

$$L_\epsilon(\phi) = -\Delta_{g^\epsilon}\phi + \epsilon h\phi - (p \pm \epsilon)\tilde{U}_\epsilon^{p \pm \epsilon - 1}\phi.$$

We shall solve the Non-Linear Problem (5.5) by using a fixed point argument based on the contraction Mapping Principle. To do so, we first establish some invertibility properties of the linear problem

$$L_\epsilon(\phi) = f \quad \text{in } \mathcal{M}_\epsilon,$$

with  $f \in L^2(\mathcal{M}_\epsilon)$ . We do this in two steps. First we study the above problem in a strip close to the scaled manifold  $K_\epsilon = \frac{K}{\sqrt{\epsilon}}$ . Let  $\gamma \in (\frac{1}{2}, 1)$  be the number fixed before in (5.3) and consider

$$\mathcal{M}_{\epsilon, \gamma} := \{x \in \mathcal{M}_\epsilon : \text{dist}_{g^\epsilon}(x, K_\epsilon) < 2\epsilon^{-\gamma}\}. \quad (5.8)$$

We are first interested in solving the following problem: given  $f \in L^2(\mathcal{M}_{\epsilon, \gamma})$

$$-\Delta_{g^\epsilon}\phi + \epsilon h\phi - (p \pm \epsilon)\tilde{U}_\epsilon^{p \pm \epsilon - 1}\phi = f \quad \text{in } \mathcal{M}_{\epsilon, \gamma}. \quad (5.9)$$

We have the validity of the following result.

**Proposition 5.1.** *There exist a constant  $C > 0$  and a sequence  $\epsilon_l = \epsilon \rightarrow 0$  such that, for any  $f \in L^2(\mathcal{M}_{\epsilon, \gamma})$  there exists a solution  $\phi \in H_{g^\epsilon}^1(\mathcal{M}_{\epsilon, \gamma})$  to Problem (5.9) such that*

$$\|\phi\|_{H_{g^\epsilon}^1} \leq C\epsilon^{-\max\{2, \frac{k}{2}\}}\|f\|_{L^2(\mathcal{M}_{\epsilon, \gamma})}. \quad (5.10)$$

The proof will be given in Section 6.

Using this, we can get the existence of solution to the linear problem in the whole domain  $\mathcal{M}_\epsilon$ .

**Proposition 5.2.** *There exist a sequence  $\epsilon_l \rightarrow 0$  and a positive constant  $C > 0$ , such that, for any  $f \in L^2(\mathcal{M}_{\epsilon_l})$ , there exists a solution  $\phi \in H^1(\mathcal{M}_{\epsilon_l})$  to the equation*

$$L_{\epsilon_l}\phi = f \quad \text{in } \mathcal{M}_{\epsilon_l}.$$

Furthermore,

$$\|\phi\|_{H_{g^\epsilon}^1(\mathcal{M}_{\epsilon_l})} \leq C\epsilon_l^{-\max\{2, \frac{k}{2}\}}\|f\|_{L^2(\mathcal{M}_{\epsilon_l})}. \quad (5.11)$$

*Proof.* By contradiction, assume that for all  $\epsilon \rightarrow 0$  there exists a solution  $(\phi_\epsilon, \lambda_\epsilon)$ ,  $\phi_\epsilon \neq 0$ , to

$$L_\epsilon(\phi_\epsilon) = \lambda_\epsilon\phi_\epsilon \quad \text{in } \mathcal{M}_\epsilon, \quad (5.12)$$

with

$$|\lambda_\epsilon|\epsilon^{-\max\{2, \frac{k}{2}\}} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (5.13)$$

Let  $\eta_\epsilon$  be a smooth cut off function (like the one defined in (5.3)) so that

$$\eta_\epsilon = 1 \quad \text{if } \text{dist}_{g^\epsilon}(z, K_\epsilon) < \frac{\epsilon^{-\gamma}}{2} \quad \text{and} \quad \eta_\epsilon = 0 \quad \text{if } \text{dist}_{g^\epsilon}(z, K_\epsilon) > \epsilon^{-\gamma}.$$

In particular one has that  $|\nabla_{K_\epsilon}\eta_\epsilon| \leq c\epsilon^\gamma$  and  $|\Delta_{K_\epsilon}\eta_\epsilon| \leq c\epsilon^{2\gamma}$  in the whole domain.

Define  $\tilde{\phi}_\epsilon = \phi_\epsilon \eta_\epsilon$ . Then  $\tilde{\phi}_\epsilon$  solves

$$L_\epsilon(\tilde{\phi}_\epsilon) = \lambda_\epsilon \tilde{\phi}_\epsilon - \nabla_{K_\epsilon} \eta_\epsilon \nabla_{K_\epsilon} a \phi_\epsilon - \Delta_{K_\epsilon} \eta_\epsilon \phi_\epsilon \text{ in } \mathcal{M}_{\epsilon, \gamma}, \quad (5.14)$$

where  $\mathcal{M}_{\epsilon, \gamma}$  is the set defined in (5.8). We now apply Proposition (5.1), that guarantees the existence of a sequence  $\epsilon_l \rightarrow 0$  and a constant  $c$  such that

$$\|\tilde{\phi}_{\epsilon_l}\|_{H_{\epsilon_l}^1} \leq c \epsilon_l^{-\max\{2, \frac{k}{2}\}} \left[ \lambda_{\epsilon_l} \|\tilde{\phi}_{\epsilon_l}\|_{L^2} + \|\nabla_{K_{\epsilon_l}} \eta_{\epsilon_l} \nabla_{K_{\epsilon_l}} \phi_{\epsilon_l}\|_{L^2} + \|\Delta_{K_{\epsilon_l}} \eta_{\epsilon_l} \phi_{\epsilon_l}\|_{L^2} \right]. \quad (5.15)$$

Observe now that, in the region where  $\nabla_{K_\epsilon} \eta_{\epsilon_l} \neq 0$  and  $\Delta_{K_\epsilon} \eta_{\epsilon_l} \neq 0$ , the function  $\tilde{U}_{\epsilon_l}$  can be uniformly bounded  $|\tilde{U}_\epsilon(y)| \leq c\epsilon$ , with a positive constant  $c$ , fact that follows directly from (5.2) and (4.8). Furthermore, since we are assuming (5.13), we see that in the region we are considering, namely where  $\nabla_{K_\epsilon} \eta_{\epsilon_l} \neq 0$  and  $\Delta_{K_\epsilon} \eta_{\epsilon_l} \neq 0$ , the function  $\phi_{\epsilon_l}$  satisfies the equation

$$-\Delta_{K_\epsilon} \phi_{\epsilon_l} + \epsilon_l^2 a_{\epsilon_l}(y) \phi_{\epsilon_l} = 0$$

for a certain smooth function  $a_{\epsilon_l}$ , which is uniformly positive and bounded as  $\epsilon_l \rightarrow 0$ . Elliptic estimates give that, in this region,  $|\phi_{\epsilon_l}| \leq c e^{-\epsilon_l^{\gamma'}}$ , and  $|\nabla_{K_\epsilon} \phi_{\epsilon_l}| \leq c e^{-\epsilon_l^{\gamma'}}$  for some  $\gamma' > 0$  and  $c > 0$ . Inserting this information in (5.15), it is easy to see that

$$\|\tilde{\phi}_{\epsilon_l}\|_{H_{\epsilon_l}^1} \leq c \epsilon_l^{-\max\{2, \frac{k}{2}\}} \lambda_{\epsilon_l} \|\tilde{\phi}_{\epsilon_l}\|_{H_{\epsilon_l}^1} (1 + o(1))$$

where  $o(1) \rightarrow 0$  as  $\epsilon_l \rightarrow 0$ . Taking into account (5.13) the above inequality gives a contradiction with the fact that, for all  $\epsilon$ , the function  $\phi_\epsilon$  is not identically zero. This concludes the proof.  $\square$

*Proof of the main Theorem.* By Proposition 5.2,  $\phi \in H_{g^\epsilon}^1(\mathcal{M}_\epsilon)$  is a solution to (5.5) if and only if

$$\phi = L_\epsilon^{-1} \left( S_\epsilon(\tilde{U}_\epsilon) + N_\epsilon(\phi) \right).$$

Notice that

$$\|N_\epsilon(\phi)\|_{L^2(\mathcal{M}_\epsilon)} \leq C \begin{cases} \|\phi\|_{H^1(\mathcal{M}_\epsilon)}^p & \text{for } p \leq 2, \\ \|\phi\|_{H^1(\mathcal{M}_\epsilon)}^2 & \text{for } p > 2 \end{cases} \quad \|\phi\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)} \leq 1 \quad (5.16)$$

and

$$\begin{aligned} & \|N_\epsilon(\phi_1) - N_\epsilon(\phi_2)\|_{L^2(\mathcal{M}_\epsilon)} \\ & \leq C \begin{cases} \left( \|\phi_1\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)}^{p-1} + \|\phi_2\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)}^{p-1} \right) \|\phi_1 - \phi_2\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)} & \text{for } p \leq 2, \\ \left( \|\phi_1\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)} + \|\phi_2\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)} \right) \|\phi_1 - \phi_2\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)} & \text{for } p > 2 \end{cases}, \end{aligned} \quad (5.17)$$

for any  $\phi_1, \phi_2$  in  $H_{g^\epsilon}^1(\mathcal{M}_\epsilon)$  with  $\|\phi_1\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)}, \|\phi_2\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)} \leq 1$ .

Defining  $T_\epsilon : H_{g^\epsilon}^1(\mathcal{M}_\epsilon) \rightarrow H_{g^\epsilon}^1(\mathcal{M}_\epsilon)$  as

$$T_\epsilon(\phi) = L_\epsilon^{-1} \left( S_\epsilon(\tilde{U}_\epsilon) + N_\epsilon(\phi) \right)$$

we will show that  $T_\epsilon$  is a contraction in some small ball in  $H_{g^\epsilon}^1(\mathcal{M}_\epsilon)$ . A direct consequence of (4.9), we have

$$\|S_\epsilon(\tilde{U}_\epsilon)\|_{L^2(\mathcal{M}_\epsilon)} \leq C \epsilon^{I+\frac{1}{2}}.$$

Using this inequality and by (5.16), (5.17) and (5.11), we obtain

$$\|T_\epsilon(\phi)\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)} \leq C \epsilon^{-\max\{2, \frac{k}{2}\}} \begin{cases} \left( \epsilon^{I+\frac{1}{2}} + \|\phi\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)}^p \right) & \text{for } p \leq 2, \\ \left( \epsilon^{I+\frac{1}{2}} + \|\phi\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)}^2 \right) & \text{for } p > 2. \end{cases}$$

Now we choose integers  $d$  and  $I$  so that

$$d > \begin{cases} \frac{\max\{2, \frac{k}{2}\}}{p-1} & \text{for } p \leq 2, \\ \max\{2, \frac{k}{2}\} & \text{for } p > 2 \end{cases} \quad I > d - \frac{1}{2} + \max\{2, \frac{k}{2}\}.$$

Thus one easily gets that  $T_\epsilon$  has a unique fixed point in set

$$\mathcal{B} = \{\phi \in H_{g^\epsilon}^1(\mathcal{M}_\epsilon) : \|\phi\|_{H_{g^\epsilon}^1(\mathcal{M}_\epsilon)} \leq \epsilon^d\},$$

as a direct application of the contraction mapping Theorem. This concludes the proof.  $\square$

## 6. A LINEAR PROBLEM: PROOF OF PROPOSITION 5.1

The quadratic functional associated to problem (5.9) given by

$$E(\phi) = \frac{1}{2} \int_{\mathcal{M}_{\epsilon, \gamma}} (|\nabla_{g^\epsilon} \phi|^2 + \epsilon h \phi^2 - (p \pm \epsilon) \tilde{U}_\epsilon^{p \pm \epsilon - 1} \phi^2) \quad (6.1)$$

for functions  $\phi \in H_{g^\epsilon}^1(\mathcal{M}_{\epsilon, \gamma})$ .

Let  $(y, x) \in \mathbb{R}^{k+N}$  be the local coordinates along  $K_\epsilon$  introduced in (1.17). With abuse of notation we will denote

$$\phi(\mathfrak{F}(y, x)) = \phi(z, x), \quad \text{with } y = \sqrt{\epsilon} z. \quad (6.2)$$

Since the original variable  $(z, x) \in \mathbb{R}^{k+N}$  are only local coordinates along  $K_\epsilon$  we let the variable  $(z, x)$  vary in the set  $\mathcal{C}_\epsilon$  defined by

$$\mathcal{C}_\epsilon = \{(z, x) / \sqrt{\epsilon} z \in K, \quad |x| < \epsilon^{-\gamma}\}. \quad (6.3)$$

We write  $\mathcal{C}_\epsilon = \frac{1}{\sqrt{\epsilon}} K \times \hat{\mathcal{C}}_\epsilon$  where

$$\hat{\mathcal{C}}_\epsilon = \{x / |x| < \epsilon^{-\gamma}\}. \quad (6.4)$$

Observe that  $\hat{\mathcal{C}}_\epsilon$  approaches, as  $\epsilon \rightarrow 0$ , the whole space  $\mathbb{R}^N$ .

In these new local coordinates, the energy density associated to the energy  $E$  in (6.1) is given by

$$\frac{1}{2} \left[ |\nabla_{g^\epsilon} \phi|^2 + \epsilon h \phi^2 - (p \pm \epsilon) \tilde{U}_\epsilon^{p \pm \epsilon - 1} \phi^2 \right] \sqrt{\det(g^\epsilon)}, \quad (6.5)$$

where  $\nabla_{g^\epsilon}$  denotes the gradient in the new variables and where  $g^\epsilon$  is the metric in the coordinates  $(z, x)$ . Using the expansions contained in the proof of Lemma 3.1, we have that, if  $(z, x)$  vary in  $\mathcal{C}_\epsilon$ , then, the energy functional (6.1) in the new variables (6.2) is given by

$$\begin{aligned} E(\phi) &= \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} \left( \frac{1}{2} (|\nabla_x \phi|^2 + \epsilon h \phi^2 - (p \pm \epsilon) \tilde{U}_\epsilon^{p \pm \epsilon - 1} \phi^2) \right) \sqrt{\det(g^\epsilon)} dz dx \\ &\quad - \frac{\epsilon}{6} \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} R_{islj} x_l x_s \partial_i \phi \partial_j \phi \sqrt{\det(g^\epsilon)} dz dx \\ &\quad + \frac{1}{2} \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} |\nabla_{K_\epsilon} \phi|^2 \sqrt{\det(g^\epsilon)} dz dx + \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} B(\phi, \phi) \sqrt{\det(g^\epsilon)} dz dx, \end{aligned} \quad (6.6)$$

where we denoted by  $B(\phi, \phi)$  a quadratic term in  $\phi$  that can be expressed in the following form

$$B(\phi, \phi) = O\left(\epsilon^{\frac{3}{2}} |x|^3\right) \partial_i \phi \partial_j \phi + \epsilon |\nabla_{K_\epsilon} \phi|^2 O(\sqrt{\epsilon} |x|) + \partial_j \phi \partial_{\bar{a}} \phi (O(\sqrt{\epsilon} |x|)) \quad (6.7)$$

and we used the Einstein convention over repeated indices. Furthermore we use the notation  $\partial_a = \partial_{y_a}$  and  $\partial_{\bar{a}} = \partial_{z_a}$ . A detailed proof of expansion (6.6) can be found in [6].

Given a function  $\phi \in H_{g^\epsilon}^1(\mathcal{M}_{\epsilon,\gamma})$ , we decompose it as

$$\phi = \left[ \frac{\delta}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) + \sum_{j=1}^N \frac{d^j}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) + \frac{e}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z) \right] \bar{\chi}_\epsilon + \phi^\perp \quad (6.8)$$

where the expression  $\tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(v)$  is defined in (5.4), the functions  $Z_0$  and  $Z_j$  are already defined in (4.15) and where  $Z$  is the eigenfunction, with  $\int_{\mathbb{R}^N} Z^2 = 1$ , corresponding to the unique positive eigenvalue  $\lambda_0$  in  $L^2(\mathbb{R}^N)$  of the problem

$$\Delta_{\mathbb{R}^N} \phi + p w_0^{p-1} \phi = \lambda_0 \phi \quad \text{in } \mathbb{R}^N. \quad (6.9)$$

It is worth mentioning that  $Z(\xi)$  is even and it has exponential decay of order  $O(e^{-\sqrt{\lambda_0}|\xi|})$  at infinity. The function  $\bar{\chi}_\epsilon$  is a smooth cut off function defined by

$$\bar{\chi}_\epsilon(x) = \hat{\chi}_\epsilon \left( \left| \left( \frac{x - \sqrt{\epsilon} \Phi_\epsilon}{\mu_\epsilon} \right) \right| \right), \quad (6.10)$$

with  $\hat{\chi}(r) = 1$  for  $r \in (0, \frac{3}{2}\epsilon^{-\gamma})$ , and  $\chi(r) = 0$  for  $r > 2\epsilon^{-\gamma}$ . Finally, in (6.8) we have that  $\delta = \delta(\sqrt{\epsilon}z)$ ,  $d^j = d^j(\sqrt{\epsilon}z)$  and  $e = e(\sqrt{\epsilon}z)$  are function defined in  $K$  such that  $\forall z \in K_\epsilon$

$$\int_{\hat{C}_\epsilon} \phi^\perp \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon dx = \int_{\hat{C}_\epsilon} \phi^\perp \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon = \int_{\hat{C}_\epsilon} \phi^\perp \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z) \bar{\chi}_\epsilon = 0. \quad (6.11)$$

We will denote by  $(H_\epsilon^1)^\perp$  the subspace of the functions in  $H_\epsilon^1$  that satisfy the orthogonality conditions (6.11).

A direct computation shows that

$$\begin{aligned} \delta(\sqrt{\epsilon}z) &= \frac{\int \phi \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0)}{\mu_\epsilon \int Z_0^2} (1 + O(\epsilon)) + O(\epsilon) \left( \sum_j d^j(\sqrt{\epsilon}z) + e(\sqrt{\epsilon}z) \right), \\ d^j(\sqrt{\epsilon}z) &= \frac{\int \phi \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_j)}{\mu_\epsilon \int Z_j^2} (1 + O(\epsilon)) + O(\epsilon) (\delta(\sqrt{\epsilon}z) + \sum_{i \neq j} d^i(\sqrt{\epsilon}z) + e(\sqrt{\epsilon}z)), \end{aligned}$$

and

$$e(\sqrt{\epsilon}z) = \frac{\int \phi \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z)}{\mu_\epsilon \int Z^2} (1 + O(\epsilon)) + O(\epsilon) (\delta(\sqrt{\epsilon}z) + \sum_j d^j(\sqrt{\epsilon}z)).$$

Observe that, since  $\phi \in H_{g^\epsilon}^1$ , one easily get that the functions  $\delta$ ,  $d^j$  and  $e$  belong to the Hilbert space

$$\mathcal{H}^1(K) = \{\zeta \in \mathcal{L}^2(K) : \partial_a \zeta \in \mathcal{L}^2(K), \quad a = 1, \dots, k\}. \quad (6.12)$$

Thanks to the above decomposition (6.8), we have the validity of the following expansion for  $E(\phi)$ .

Observe that in the region we are considering the function  $\tilde{U}_\epsilon$  is nothing but  $\tilde{U}_\epsilon = \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(v_{I+1, \epsilon})$ , where  $v_{I+1, \epsilon}$  is the function whose existence and properties are proven in Lemma 4.1. For the argument in this part of our proof it is enough to take  $I = 3$ , and for simplicity of notation we will denote by  $\hat{w}$  the function  $v_{I+1, \epsilon}$  with  $I = 3$ . Referring to (4.9) we have

$$\hat{w}(z, \xi) = w_0(\xi) + \sum_{i=1}^4 w_{i, \epsilon}(z, \xi) \quad (6.13)$$

where  $w_0$  is defined by (4.12) and

$$\|D_\xi^2 w_{i+1, \epsilon}\|_{\epsilon, N-2, \sigma} + \|D_\xi w_{i+1, \epsilon}\|_{\epsilon, N-3, \sigma} + \|w_{i+1, \epsilon}\|_{\epsilon, N-4, \sigma} \leq C \epsilon^{i+\frac{1}{2}} \quad (6.14)$$

and, for any integer  $\ell$

$$\|\nabla_y^{(\ell)} w_{i+1,\epsilon}(y, \cdot)\|_{\epsilon, N-2, \sigma} \leq \beta C_l \epsilon^{i+\frac{1}{2}} \quad y = \sqrt{\epsilon} z \in K$$

for any  $i = 0, 1, 2, 3$ .

**Theorem 6.1.** *Let  $\gamma = 1 - \sigma$ , for some  $\sigma > 0$  and small. Assume we write  $\phi \in H_\epsilon^1$  as in (6.8) and let  $d = (d^1, \dots, d^N)$ . Then, there exists  $\epsilon_0 > 0$  such that, for all  $0 < \epsilon < \epsilon_0$ , the following expansion holds true*

$$E(\phi) = E(\phi^\perp) + \epsilon^{-\frac{k}{2}} [P_\epsilon(\delta) + Q_\epsilon(d) + R_\epsilon(e)] + \mathcal{G}(\phi^\perp, \delta, d, e). \quad (6.15)$$

In (6.15)

$$P_\epsilon(\delta) = P(\delta) + P_1(\delta) \quad (6.16)$$

with

$$\begin{aligned} P(\delta) &= \frac{A_\epsilon}{2} \int_K \epsilon |\nabla_K(\delta(1 + o(\epsilon)\beta_1^\epsilon(y)))|^2 \\ &\quad + \epsilon \int_K \delta^2 \left( 2c_{3,N}h - c_{2,N} \sum_j \left[ \sum_s \frac{2}{3} R_{mssj} + \sum_{m,a,b} (\tilde{g}^{ab} R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \right] \right) \\ &\quad \mp \epsilon c_{4,N} \int_K \frac{\delta^2}{\mu_0} \end{aligned}$$

where  $A_\epsilon$  a real number such that  $\lim_{\epsilon \rightarrow 0} A_\epsilon = c_{1,N} := \int_{\mathbb{R}^N} Z_0^2$ , and  $c_{2,N}, c_{3,N}, c_{4,N}$  are given in (4.30)-(4.32),  $\beta_1^\epsilon$  is an explicit smooth function defined on  $K$  which is uniformly bounded as  $\epsilon \rightarrow 0$ ; furthermore,  $P_1(\delta)$  is a small compact perturbation in  $\mathcal{H}_g^1(K)$  whose shape is a sum of quadratic functional in  $\delta$  of the form

$$\epsilon^2 \int_K b(y) |\delta|^2$$

where  $b(y)$  denotes a generic explicit function, smooth and uniformly bounded, as  $\epsilon \rightarrow 0$ , in  $K$ . In (6.15),

$$Q_\epsilon(d) = Q(d) + Q_1(d) \quad (6.17)$$

with

$$Q(d) = \frac{\epsilon}{2} C_\epsilon \left( \int_K |\nabla_K(d(1 + o(\epsilon^2)\beta_2^\epsilon(y)))|^2 + \int_K (\tilde{g}^{ab} R_{mabl} - \Gamma_a^c(E_m) \Gamma_c^a(E_l)) d^m d^l \right) \quad (6.18)$$

where  $C_\epsilon$  is a real number such that  $\lim_{\epsilon \rightarrow 0} C_\epsilon = C := \int_{\mathbb{R}_+^N} Z_1^2$ ,  $\beta_2^\epsilon$  is an explicit smooth function defined on  $K$  which is uniformly bounded as  $\epsilon \rightarrow 0$  and the terms  $R_{maal}$  and  $\Gamma_a^c(E_m)$  are smooth functions on  $K$ . Furthermore,  $Q_1(d)$  is a small compact perturbation in  $\mathcal{H}^1(K)$  whose shape is a sum of quadratic functional in  $d$  of the form

$$\epsilon^3 \int_K b(y) d^i d^j$$

where again  $b(y)$  is a generic explicit function, smooth and uniformly bounded, as  $\epsilon \rightarrow 0$ , in  $K$ . In (6.15),

$$R_\epsilon(e) = R(e) + R_1(e) \quad (6.19)$$

$$R(e) = \epsilon^{-\frac{k}{2}} \left[ \frac{D_\epsilon}{2} \left( \epsilon^2 \int_K |\nabla_K(e(1 + e^{-\frac{\lambda_0}{2}\epsilon^{-\gamma}} \beta_3^\epsilon(y)))|^2 - \lambda_0 \int_K e^2 \right) \right] \quad (6.20)$$

with  $D_\epsilon$  a real number so that  $\lim_{\epsilon \rightarrow 0} D_\epsilon = D := \int_{\mathbb{R}^N} Z^2$ ,  $\beta_3^\epsilon$  an explicit smooth function in  $K$ , which is uniformly bounded as  $\epsilon \rightarrow 0$ , and  $\lambda_0$  the positive number defined in (6.9). Furthermore,

$R_1$  is a small compact perturbation in  $\mathcal{H}^1(K)$  whose shape is a sum of quadratic functional in  $e$  of the form

$$\epsilon^3 \int_K b(y) e^2$$

where again  $b(y)$  is a generic explicit function, smooth and uniformly bounded, as  $\epsilon \rightarrow 0$ , in  $K$ . Finally in (6.15)

$$\mathcal{G} : (H_{g^\epsilon}^1)^\perp \times (\mathcal{H}^1(K))^{N+1} \rightarrow \mathbb{R}$$

is a continuous and differentiable functional with respect to the natural topologies, homogeneous of degree 2

$$\mathcal{G}(t\phi^\perp, t\delta, td, te) = t^2 \mathcal{M}(\phi^\perp, \delta, d, e) \quad \forall t.$$

The derivative of  $\mathcal{G}$  with respect to each one of its variable is given by a small multiple of a linear operator in  $(\phi^\perp, \delta, d, e)$  and it satisfies

$$\begin{aligned} & \|D_{(\phi^\perp, \delta, d)} \mathcal{G}(\phi_1^\perp, \delta_1, d_1, e_1) - D_{(\phi^\perp, \delta, d)} \mathcal{G}(\phi_2^\perp, \delta_2, d_2, e_2)\| \leq C\epsilon^{\gamma(N-3)} \times \\ & \left[ \|\phi_1^\perp - \phi_2^\perp\| + \epsilon^{-\frac{k}{2}} \|\delta_1 - \delta_2\|_{\mathcal{H}^1(K)} + \epsilon^{-\frac{k}{2}} \|d_1 - d_2\|_{(\mathcal{H}^1(K))^{N-1}} + \epsilon^{-\frac{k}{2}} \|e_1 - e_2\|_{\mathcal{H}^1(K)} \right]. \end{aligned} \quad (6.21)$$

Furthermore, there exists a constant  $C > 0$  such that

$$|\mathcal{G}(\phi^\perp, \delta, d, e)| \leq C\epsilon^2 \left[ \|\phi^\perp\|^2 + \epsilon^{-\frac{k}{2}} \left( \|\delta\|_{\mathcal{H}^1(K)}^2 + \|d\|_{\mathcal{H}^1(K)}^2 + \|e\|_{\mathcal{H}^1(K)}^2 \right) \right]. \quad (6.22)$$

*Proof.* STEP 1. We claim that there exists  $\epsilon_0 > 0$  such that for all  $0 < \epsilon < \epsilon_0$ , we have

$$E\left(\frac{\delta}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon\right) = \epsilon^{-\frac{k}{2}} P_\epsilon(\delta), \quad (6.23)$$

$$E\left(\frac{d^j}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon\right) = \epsilon^{-\frac{k}{2}} Q_\epsilon(d^j), \quad (6.24)$$

$$E\left(\frac{e}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z) \bar{\chi}_\epsilon\right) = \epsilon^{-\frac{k}{2}} R_\epsilon(e). \quad (6.25)$$

Define

$$\begin{aligned} F(u) : &= \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} \left( \frac{1}{2} |\nabla_x u|^2 + \frac{1}{2} \epsilon h u^2 - \frac{1}{p+1 \pm \epsilon} u^{p+1 \pm \epsilon} \right) \sqrt{\det(g^\epsilon)} dz dx \\ &\quad - \frac{\epsilon}{6} \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} R_{islj} x_l x_s \partial_i u \partial_j u \sqrt{\det(g^\epsilon)} dz dx \\ &\quad + \frac{1}{2} \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} |\nabla_{K_\epsilon} u|^2 \sqrt{\det(g^\epsilon)} dz dx + \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} B(u, u) \sqrt{\det(g^\epsilon)} dz dx. \end{aligned} \quad (6.26)$$

To prove (6.23), we write for small  $t \neq 0$

$$\begin{aligned} & \left[ DF(\tilde{\mathcal{T}}_{\mu_\epsilon + t\delta, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon) - DF(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon) \right] \left( \frac{\delta}{\mu_\epsilon + t\delta} \tilde{\mathcal{T}}_{\mu_\epsilon + t\delta, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \right) \\ &= -2tE\left(\frac{\delta}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon\right)(1 + O(t)). \end{aligned} \quad (6.27)$$

On the other hand, for any  $\psi$

$$\left[ DF(\tilde{\mathcal{T}}_{\mu_\epsilon + t\delta, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon) - DF(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon) \right] (\psi) = \mathbf{a}(t) - \mathbf{a}(0) + \mathbf{b}(t) + \mathbf{c}(t) \quad (6.28)$$

where

$$\begin{aligned}
\mathbf{a}(t) &= \int_{K_\epsilon \times \hat{C}_\epsilon} \left[ \left( \nabla_x \tilde{T}_{\mu_\epsilon+t\delta, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon \right) \nabla_x \psi + \epsilon h \tilde{T}_{\mu_\epsilon+t\delta, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon \psi \right. \\
&\quad \left. - \left( \tilde{T}_{\mu_\epsilon+t\delta, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon \right)^{p^{\pm\epsilon}} \psi \right] \sqrt{\det(g^\epsilon)} dz dx \\
&\quad - \frac{\epsilon}{6} \int_{K_\epsilon \times \hat{C}_\epsilon} R_{islj} x_l x_s \partial_i \left( \tilde{T}_{\mu_\epsilon+t\delta, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon \right) \partial_j \psi \sqrt{\det(g^\epsilon)} dz dx, \\
\mathbf{b}(t) &= \int_{K_\epsilon \times \hat{C}_\epsilon} \partial_{\bar{a}}(\tilde{T}_{\mu_\epsilon+t\delta, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon) \partial_{\bar{a}} \psi \sqrt{\det(g^\epsilon)} dz dx \\
&\quad - \int_{K_\epsilon \times \hat{C}_\epsilon} \partial_{\bar{a}}(\tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon) \partial_{\bar{a}} \psi \sqrt{\det(g^\epsilon)} dz dx
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{c}(t) &= \int_{K_\epsilon \times \hat{C}_\epsilon} B(\tilde{T}_{\mu_\epsilon+t\delta, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon, \psi) \sqrt{\det(g^\epsilon)} dz dx \\
&\quad - \int_{K_\epsilon \times \hat{C}_\epsilon} B(\tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w}) \bar{\chi}_\epsilon, \psi) \sqrt{\det(g^\epsilon)} dz dx.
\end{aligned}$$

We now compute  $\mathbf{a}(t) - \mathbf{a}(0)$  with  $\psi = \frac{\delta}{\mu_\epsilon+t\delta} \tilde{T}_{\mu_\epsilon+t\delta, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon$ . Performing the change of variables  $x = (\mu_\epsilon + t\delta)\xi + \sqrt{\epsilon}\Phi_\epsilon$  and using the expansion of  $\frac{\delta}{\mu_\epsilon+t\delta} = \frac{\delta}{\mu_\epsilon} - \frac{\delta^2}{\mu_\epsilon^2}t + O(t^2)$ , we see that

$$\begin{aligned}
&t^{-1} [\mathbf{a}(t) - \mathbf{a}(0)] \\
&= -\epsilon \left\{ \int_{K_\epsilon} \delta^2 \left( c_{3,N} h - c_{2,N} \sum_j \left[ \sum_s \frac{2}{3} R_{mssj} + \sum_{m,a,b} (\tilde{g}^{ab} R_{mabj} - \Gamma_{am}^b \Gamma_{bj}^a) \right] \right) \mp c_{4,N} \int_{K_\epsilon} \frac{\delta^2}{\mu_\epsilon^2} \right\} \\
&\quad \times (1 + O(t)) \left( 1 + O(\epsilon^{\gamma(N-4)}) \right).
\end{aligned}$$

On the other hand, by the definition of the function  $\mathbf{b}(t)$  above, a Taylor expansion gives

$$\mathbf{b}(t) = -t \left( \int_{K_\epsilon \times \hat{C}_\epsilon} |\nabla_{K_\epsilon} \left( \frac{\delta}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \right)|^2 dz dx \right) \times (1 + O(t)).$$

Observe now that

$$\begin{aligned}
\partial_{\bar{a}} \left( \frac{\delta}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \right) &= (\partial_{\bar{a}} \delta) \frac{1}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon + \delta \partial_{\bar{a}} \left( \frac{1}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \right) \\
&= \sqrt{\epsilon} (\partial_{\bar{a}} \delta) \frac{1}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon + \sqrt{\epsilon} \delta (\partial_{\bar{a}} \mu_\epsilon) \partial_{\mu_\epsilon} \left( \frac{1}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \right) \\
&\quad + \sqrt{\epsilon} \delta (\partial_{\bar{a}} \Phi_\epsilon) \partial_{\Phi_\epsilon} \left( \frac{1}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \right).
\end{aligned}$$

Since  $\int \left( \frac{1}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \right)^2 dx = \int \left( \frac{1}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \right)^2 dx = c_{1,N} (1 + o(\epsilon))$ , we conclude that

$$\mathbf{b}(t) = -t \epsilon^{-\frac{k}{2}} \left[ A_\epsilon \epsilon \int_K |\nabla_K (\delta(1 + o(\epsilon)) \beta_1^\epsilon(y))|^2 \right] \quad (6.29)$$

where  $A_\epsilon \in \mathbb{R}$ ,  $\lim_{\epsilon \rightarrow 0} A_\epsilon = c_{1,N} = \int_{\mathbb{R}^N} Z_0^2$  and  $\beta_1^\epsilon$  is an explicit smooth function in  $K$ , which is uniformly bounded as  $\epsilon \rightarrow 0$ . Finally we observe that the last term  $\mathbf{c}(t)$  defined above is of lower order, and can be absorbed in the terms already described.

Proof of (6.24). Let  $d$  be the vector field along  $K$  defined by

$$d(\sqrt{\epsilon}z) = (d^1(\sqrt{\epsilon}z), \dots, d^N(\sqrt{\epsilon}z)).$$

For any  $t$  small and  $t \neq 0$ , we have (see (6.26))

$$\begin{aligned} & \left[ DF(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon) - DF(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w})\bar{\chi}_\epsilon) \right] [\varphi] \\ &= tD^2F(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w})\bar{\chi}_\epsilon) \left[ \sum_l \frac{d^l}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_l)\bar{\chi}_\epsilon \right] [\varphi] (1 + O(t))(1 + O(\epsilon)) \end{aligned}$$

for any function  $\varphi \in H_{g^\epsilon}^1$ . Choosing  $\varphi = \frac{d^j}{\sqrt{\epsilon}\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j)\bar{\chi}_\epsilon$  we write

$$\begin{aligned} & \left[ DF(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon) - DF(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w})\bar{\chi}_\epsilon) \right] \left[ \frac{d^j}{\sqrt{\epsilon}\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j)\bar{\chi}_\epsilon \right] \\ &= \mathbf{a}_2(t) - \mathbf{a}_2(0) + \mathbf{b}_2(t) + \mathbf{c}_2(t) \end{aligned} \quad (6.30)$$

where we have set, for  $\psi = \frac{d^j}{\sqrt{\epsilon}\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j)\bar{\chi}_\epsilon$ ,

$$\begin{aligned} \mathbf{a}_2(t) &= \int \left( \nabla_X \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon \right) \nabla_X \psi + \epsilon \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon \psi - \left( \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon \right)^{p \pm \epsilon} \psi \\ &\quad - \frac{\epsilon}{6} \int R_{islj} x_l x_s \partial_i \left( \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon \right) \partial_j \psi, \end{aligned}$$

$$\mathbf{b}_2(t) = \int \partial_{\bar{a}}(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon) \partial_{\bar{a}}(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon) - \int \partial_{\bar{a}}(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w})\bar{\chi}_\epsilon) \partial_{\bar{a}}(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w})\bar{\chi}_\epsilon)$$

and

$$\mathbf{c}_2(t) = \int B(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon + td}(\hat{w})\bar{\chi}_\epsilon, \psi) - \int B(\tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(\hat{w})\bar{\chi}_\epsilon, \psi).$$

Define

$$\mathcal{R}_{ml} = \left( (\hat{g})^{ab} R_{mabl} - \Gamma_a^c(E_m) \Gamma_c^a(E_l) \right).$$

Performing the change of variables  $x = \mu_\epsilon \xi + \sqrt{\epsilon}(\Phi_\epsilon + td)$ , we get

$$\begin{aligned} & t^{-1}[\mathbf{a}_2(t) - \mathbf{a}_2(0)] \\ &= \epsilon \left\{ \int \frac{d^j}{\mu_\epsilon} \left[ \nabla \hat{w} \nabla Z_j + \epsilon \mu_\epsilon^2 h \hat{w} - \mu_\epsilon^{\mp \frac{N-2}{2}} \epsilon \hat{w}^{p \pm \epsilon} Z_j \right] \right. \\ &\quad \times \left( \frac{R_{mijl}}{6} + \frac{\mathcal{R}_{lm}}{2} \right) [(\mu_\epsilon \xi_m + \sqrt{\epsilon} \Phi_{\epsilon, m}) d^l + (\mu_\epsilon \xi_l + \sqrt{\epsilon} \Phi_{\epsilon, l}) d^m] \\ &\quad \left. - \int \frac{d^j}{\mu_\epsilon} \frac{R_{ilsr}}{6} [(\mu_\epsilon \xi_s + \sqrt{\epsilon} \Phi_{\epsilon, s}) d^l + (\mu_\epsilon \xi_l + \sqrt{\epsilon} \Phi_{\epsilon, l}) d^s] \partial_i \hat{w} \partial_r Z_j \right\} \\ &\quad \times (1 + O(\epsilon))(1 + O(t)). \end{aligned}$$

Integration by parts in the  $\xi$  variables, using the fact that  $\hat{\mathcal{C}}_\epsilon \rightarrow \mathbb{R}^N$  as  $\epsilon \rightarrow 0$ ,  $R_{irll} = 0$ , we deduce that

$$\begin{aligned} t^{-1}[\mathbf{a}_2(t) - \mathbf{a}_2(0)] &= \epsilon \left\{ -C \int_{K_\epsilon} \left( \frac{R_{miij}}{6} + \frac{\mathcal{R}_{mj}}{2} \right) d^j d^m + C \int_{K_\epsilon} \frac{R_{jrrm}}{3} d^m d^j \right\} \times \\ &\quad \times (1 + O(\epsilon)) (1 + O(t)) \\ &= \epsilon^{-\frac{k}{2}} \left[ -C \int \frac{\mathcal{R}_{mj}}{2} d^j d^m + O(\epsilon) Q(d) \right] (1 + O(t)) \end{aligned} \quad (6.31)$$



where here we have set

$$C = \int_{\mathbb{R}^N} Z_1^2 \quad \text{and} \quad Q(d) := \int_K \pi(y) d^i d^j$$

for some smooth and uniformly bounded (as  $\epsilon \rightarrow 0$ ) function  $\pi(y)$ . To estimate the term  $\mathfrak{b}_2$  above we argue as in (6.29), we get that

$$t^{-1} \mathfrak{b}_2(t) = -\epsilon^{-\frac{k}{2}} \left[ \epsilon C_\epsilon \int_K |\nabla_K(d^j(1 + \beta_2^\epsilon(y) o(\epsilon)))|^2 \right] (1 + O(t)). \quad (6.32)$$

Finally we observe that the last term  $\mathfrak{c}_2(t)$  is of lower order, and can be absorbed in the terms described in (6.31) and (6.32).

Proof of (6.25). To get the expansion in (6.25), we compute

$$E\left(\frac{e}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z)\right) = I + II + III \quad (6.33)$$

where

$$\begin{aligned} I &= \int_{K_\epsilon \times \hat{C}_\epsilon} \frac{e^2}{\mu_\epsilon^2} \left( \frac{1}{2} (|\nabla_x \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z)|^2 + \epsilon h \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z)^2 - (p \pm \epsilon) \tilde{U}_\epsilon^{p \pm \epsilon - 1} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z)^2) \right) \times \\ &\quad \sqrt{\det g^\epsilon} dz dx \\ &\quad - \frac{\epsilon}{6} \int_{K_\epsilon \times \hat{C}_\epsilon} \frac{e^2}{\mu_\epsilon^2} R_{islj} x_s x_l \partial_i \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z) \partial_j \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z) \sqrt{\det g^\epsilon} dz dx, \\ II &= \frac{1}{2} \int_{K_\epsilon \times \hat{C}_\epsilon} |\nabla_K \left( \frac{e}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z) \right)|^2 \sqrt{\det g^\epsilon} dz dx \end{aligned}$$

and

$$III = \int_{K_\epsilon \times \hat{C}_\epsilon} B\left(\frac{e}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z), \frac{e}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z)\right) \sqrt{\det(g^\epsilon)} dz dx.$$

Using the change of variables  $x = \mu_\epsilon \xi + \sqrt{\epsilon} \Phi_\epsilon$  in  $I$ , we have

$$I = \int \frac{1}{2} \frac{e^2}{\mu_\epsilon^2} \left[ |\nabla Z|^2 - p \hat{w}^{p-1} Z^2 + \epsilon \mu_\epsilon^2 h Z^2 \right] \left( 1 + \epsilon O(e^{-|\xi|}) \right).$$

Then, recalling the definition of  $\lambda_0$  in (6.9), we get

$$I = \epsilon^{-\frac{k}{2}} \left[ -\frac{\lambda_0}{2} D \int_K e^2 + \epsilon Q(e) \right] \quad (6.34)$$

where we have set

$$D = \int_{\mathbb{R}^N} Z^2(\xi) d\xi \quad \text{and} \quad Q(e) := \int_K \tau(y) e^2 dy,$$

for some smooth and uniformly bounded, as  $\epsilon \rightarrow 0$ , function  $\tau$ . On the other hand, using a direct computation and arguing as in (6.29), we get

$$II = \frac{D_\epsilon}{2} \int_{K_\epsilon} |\nabla_{K_\epsilon} e + e^{-\lambda_0 \epsilon^{-\gamma}} \beta_3^\epsilon(\epsilon z) e|^2 = \epsilon^{-\frac{k}{2}} \left[ \frac{D_\epsilon}{2} \epsilon \int_K |\nabla_K (e(1 + e^{-\lambda' \epsilon^{-\gamma}} \beta_3^\epsilon(y)))|^2 \right] \quad (6.35)$$

where  $\beta_3^\epsilon$  is an explicit smooth function on  $K$ , which is uniformly bounded as  $\epsilon \rightarrow 0$ , while  $\lambda'$  is a positive real number. Finally we observe that the last term  $III$  is of lower order, and can be absorbed in the terms described in (6.34) and (6.35). This concludes the proof of (6.25).

STEP 2. We write

$$\begin{aligned} \mathcal{G}(\phi^\perp, \delta, d, e) &= E(\phi) - E(\phi^\perp) - E\left(\frac{\delta}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon\right) - \sum_{j=1}^N E\left(\frac{d_j}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon\right) \\ &\quad - E\left(\frac{e}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z) \bar{\chi}_\epsilon\right). \end{aligned}$$

Thus it is clear that the term  $\mathcal{G}$  recollects all the mixed terms in the expansion of  $E(\phi)$ . Indeed, if we define

$$\begin{aligned} m(f_1, f_2) &= \int_{K_\epsilon \times \hat{C}_\epsilon} (\nabla_x f_1 \nabla_x f_2 - (p \pm \epsilon) U_\epsilon^{p \pm \epsilon - 1} f_1 f_2) \sqrt{\det(g^\epsilon)} dz dx \\ &\quad + \epsilon \int_{K_\epsilon \times \hat{C}_\epsilon} h f_1 f_2 dz dx \\ &\quad - \frac{\epsilon}{6} \int_{K_\epsilon \times \hat{C}_\epsilon} R_{islj} x_s x_l \partial_i f_1 \partial_j f_2 \sqrt{\det(g^\epsilon)} dz dx \\ &\quad + \int_{K_\epsilon \times \hat{C}_\epsilon} \partial_{\bar{a}} f_1 \partial_{\bar{a}} f_2 \sqrt{\det(g^\epsilon)} dz dx + \int_{K_\epsilon \times \hat{C}_\epsilon} B(f_1, f_2) \sqrt{\det(g^\epsilon)} dz dx \\ &:= m_1(f_1, f_2) + m_2(f_1, f_2) + m_3(f_1, f_2) + m_4(f_1, f_2) + m_5(f_1, f_2), \end{aligned}$$

for  $f_1$  and  $f_2$  in  $H_\epsilon^1$ , then

$$\begin{aligned} \mathcal{G}(\phi^\perp, \delta, d, e) &= m(\phi^\perp, \frac{\delta}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon) + \sum_j m(\phi^\perp, \frac{d_j}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon) \\ &\quad + m(\phi^\perp, \frac{e}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z) \bar{\chi}_\epsilon) + \sum_j m(\frac{\delta}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon, \frac{d_j}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon) \\ &\quad + \sum_{i \neq j} m(\frac{d_j}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon, \frac{d_i}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_i) \bar{\chi}_\epsilon) \\ &\quad + m(\frac{\delta}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon, \frac{e}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z) \bar{\chi}_\epsilon) \\ &\quad + \sum_j m(\frac{d_j}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon, \frac{e}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z) \bar{\chi}_\epsilon). \end{aligned} \tag{6.36}$$

One can see clearly that  $\mathcal{G}$  is homogeneous of degree 2 and that its first derivatives with respect to its variables is a linear operator in  $(\phi^\perp, \delta, d, e)$ . We will then show the validity of estimate (6.22). In a very similar way one shows the validity of (6.21). To prove (6.22), we should treat each one of the above terms. Since the computations are very similar, we will limit ourselves to treat the term

$$m := m\left(\frac{\delta}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon, \frac{d_j}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon\right).$$

This term can be written as

$$m = \sum_{i=1}^5 m_i(f_1, f_2) \tag{6.37}$$

with  $f_1 = \frac{\delta}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon$  and  $f_2 = \frac{d_j}{\mu_\epsilon} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon$ . Using the fact that the function  $Z_0$  solves

$$\Delta Z_0 + p w_0^{p-1} Z_0 = 0 \quad \text{in } \mathbb{R}^N,$$

with  $\int_{\mathbb{R}^N} \partial_{\xi_N} Z_0 Z_j = 0$  and integrating by parts in the  $x$  variable (recalling the expansion of  $\sqrt{\det g^\epsilon}$ ), one gets

$$\begin{aligned} m_1 &= \left\{ \int \frac{\delta d^j}{\mu_\epsilon^2} [-\Delta \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) - (p \pm \epsilon) U_\epsilon^{p \pm \epsilon - 1} \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0)] \tilde{\chi}_\epsilon^2 \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \sqrt{\det g^\epsilon} \right. \\ &\quad \left. + \int \frac{\delta d^j}{\mu_\epsilon^2} \partial_{\xi_N} \left( \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \tilde{\chi}_\epsilon \right) \tilde{\mathcal{T}}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \frac{1}{\mu_\epsilon} (\epsilon \operatorname{tr}(H) + O(\epsilon^2)) \tilde{\chi}_\epsilon \right\} (1 + o(1)) \end{aligned}$$

where  $o(1) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Thus, a Hölder inequality yields

$$|m_1| \leq C \epsilon^{-\frac{k}{2}} \epsilon^{\gamma(N-2)} \|\delta\|_{\mathcal{L}^2(K)} \|d^j\|_{\mathcal{L}^2(K)}.$$

On the other hand, using the orthogonality condition  $\int_{\mathbb{R}^N} Z_0 Z_j = 0$ , we get

$$|m_2| \leq C \epsilon \epsilon^{-\frac{k}{2}} \left( \int_{|\xi| > \epsilon^{-\gamma}} Z_0 Z_j \right) \|\delta\|_{\mathcal{L}^2(K)} \|d^j\|_{\mathcal{L}^2(K)} \leq C \epsilon^{-k} \epsilon^{1+\gamma(N-3)} \|\delta\|_{\mathcal{L}^2(K)} \|d^j\|_{\mathcal{L}^2(K)}.$$

Now, since  $\int_{\mathbb{R}^N} \xi_N \partial_i Z_0 \partial_l Z_j = 0$ , for any  $i, j, l = 1, \dots, N-1$ , one gets

$$\begin{aligned} |m_3| &\leq C \epsilon \epsilon^{-\frac{k}{2}} \left( \int_{|\xi| > \epsilon^{-\gamma}} \xi_N \partial_i Z_0 \partial_l Z_j \right) \|\delta\|_{\mathcal{L}^2(K)} \|d^j\|_{\mathcal{L}^2(K)} \\ &\leq C \epsilon^{-\frac{k}{2}} \epsilon^{1+\gamma(N-2)} \|\delta\|_{\mathcal{L}^2(K)} \|d^j\|_{\mathcal{L}^2(K)}. \end{aligned}$$

A direct computation on the term  $m_4$  gives

$$\begin{aligned} |m_4| &\leq C \epsilon^{-\frac{k}{2}} \left\{ \epsilon^2 \left( \int_{|\xi| > \epsilon^{-\gamma}} Z_0 Z_j \right) \|\partial_a \delta\|_{\mathcal{L}^2(K)} \|\partial_a d^j\|_{\mathcal{L}^2(K)} \right. \\ &\quad + \epsilon \left( \int_{|\xi| > \epsilon^{-\gamma}} Z_0 Z_j \right) (\|\delta\|_{\mathcal{L}^2(K)} \|\partial_a d^j\|_{\mathcal{L}^2(K)} + \|\partial_a \delta\|_{\mathcal{L}^2(K)} \|d^j\|_{\mathcal{L}^2(K)}) \\ &\quad \left. + \left( \int_{|\xi| > \epsilon^{-\gamma}} Z_0 Z_j \right) \|\delta\|_{\mathcal{L}^2(K)} \|d^j\|_{\mathcal{L}^2(K)} \right\} \\ &\leq C \epsilon^{-\frac{k}{2}} \epsilon^{\gamma(N-3)} [\|\delta\|_{\mathcal{H}^1(K)}^2 + \|d^j\|_{\mathcal{H}^1(K)}^2]. \end{aligned}$$

Since  $|m_5| \leq C \sum_{j=1}^4 |m_j|$  we conclude that

$$|m| \leq C \epsilon^{-\frac{k}{2}} \epsilon^{\gamma(N-3)} [\|\delta\|_{\mathcal{H}^1(K)}^2 + \|d^j\|_{\mathcal{H}^1(K)}^2].$$

Each one of the terms appearing in (6.36) can be estimated to finally get the validity of (6.22). This conclude the proof of Theorem (6.1).  $\square$

Now, we are going to prove Proposition 5.1.

*Proof.* We define the energy functional associated to Problem (5.9)

$$\mathcal{E} : (H_{g^\epsilon}^1)^\perp \times (\mathcal{H}^1(K))^{N+2} \rightarrow \mathbb{R}$$

by

$$\mathcal{E}(\phi^\perp, \delta, d, e) = E(\phi) - \mathcal{L}_f(\phi) \tag{6.38}$$

where  $E$  is the functional in (6.1) and  $\mathcal{L}_f(\phi)$  is the linear operator given by

$$\mathcal{L}_f(\phi) = \int_{\mathcal{M}_{\epsilon, \gamma}} f \phi.$$

Observe that

$$\mathcal{L}_f(\phi) = \mathcal{L}_f^1(\phi^\perp) + \epsilon^{-\frac{k}{2}} [\mathcal{L}_f^2(\delta) + \mathcal{L}_f^3(d) + \mathcal{L}_f^4(e)]$$

where  $\mathcal{L}_f^1 : H_{g^\epsilon}^1 \rightarrow \mathbb{R}$ ,  $\mathcal{L}_f^2, \mathcal{L}_f^4 : \mathcal{H}^1(K) \rightarrow \mathbb{R}$  and  $\mathcal{L}_f^3 : (\mathcal{H}^1(K))^N \rightarrow \mathbb{R}$  with

$$\begin{aligned} \mathcal{L}_f^1(\phi^\perp) &= \int_{\mathcal{M}_{\epsilon, \gamma}} f \phi^\perp, \quad \epsilon^{-\frac{k}{2}} \mathcal{L}_f^2(\delta) = \int_{\mathcal{M}_{\epsilon, \gamma}} f \frac{\delta}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_0) \bar{\chi}_\epsilon \\ \epsilon^{-\frac{k}{2}} \mathcal{L}_f^3(d) &= \sum_{j=1}^N \int_{\mathcal{M}_{\epsilon, \gamma}} f \frac{d^j}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z_j) \bar{\chi}_\epsilon \quad \text{and} \quad \epsilon^{-\frac{k}{2}} \mathcal{L}_f^4(e) = \int_{\mathcal{M}_{\epsilon, \gamma}} f \frac{e}{\mu_\epsilon} \tilde{T}_{\mu_\epsilon, \Phi_\epsilon}(Z) \bar{\chi}_\epsilon. \end{aligned}$$

Finding a solution  $\phi \in H_{g^\epsilon}^1$  to Problem (5.9) reduces to finding a critical point  $(\phi^\perp, \delta, d, e)$  for  $\mathcal{E}$ . This will be done in several steps.

**Step 1.** We claim that there exist  $\sigma > 0$  and  $\epsilon_0$  such that for all  $\epsilon \in (0, \epsilon_0)$  and all  $\phi^\perp \in (H_{g^\epsilon}^1)^\perp$  then

$$E(\phi^\perp) \geq \sigma \|\phi^\perp\|_{L^2}^2. \quad (6.39)$$

In fact, using the local change of variables (6.2), together with the expansion of energy  $E$  in (6.6), we see that, for sufficiently small  $\epsilon > 0$

$$E(\phi^\perp) \geq \frac{1}{4} E_0(\phi^\perp),$$

with

$$E_0(\phi^\perp) = \int_{K_\epsilon \times \hat{\mathcal{C}}_\epsilon} [|\nabla_x \phi^\perp|^2 - (p \pm \epsilon) U_\epsilon^{p \pm \epsilon - 1}(\phi^\perp)^2] \sqrt{\det(g^\epsilon)} dz dx$$

for any  $\phi^\perp = \phi^\perp(\sqrt{\epsilon}z, x)$ , with  $z \in K_\epsilon = \frac{1}{\sqrt{\epsilon}}K$ . The set  $\hat{\mathcal{C}}_\epsilon$  is defined in (6.4) and the function  $U_\epsilon$  is given by (5.2). We recall that  $\hat{\mathcal{C}}_\epsilon \rightarrow \mathbb{R}^N$  as  $\epsilon \rightarrow 0$ .

We will establish (6.39) showing that

$$E_0(\phi^\perp) \geq \sigma \|\phi^\perp\|_{L^2}^2 \quad \forall \phi^\perp. \quad (6.40)$$

To do so, we first observe that if we scale in the  $z$ -variable, defining  $\varphi^\perp(y, x) = \phi^\perp(\frac{y}{\sqrt{\epsilon}}, x)$ , the relation (6.40) becomes

$$E_0(\varphi^\perp) \geq \sigma \|\varphi^\perp\|_{L^2}^2. \quad (6.41)$$

Thus we are led to show the validity of (6.41). We argue by contradiction. Assume that for any  $n$ , there exist  $\epsilon_n \rightarrow 0$  and  $\varphi_n^\perp \in (H_{g^{\epsilon_n}}^1)^\perp$  such that

$$E_0(\varphi_n^\perp) \leq \frac{1}{n} \|\varphi_n^\perp\|_{L^2}^2. \quad (6.42)$$

Without loss of generality we can assume that the sequence  $(\|\varphi_n^\perp\|)_n$  is bounded, as  $n \rightarrow \infty$ . Hence, up to subsequences, we have that

$$\varphi_n^\perp \rightharpoonup \varphi^\perp \quad \text{in } H^1(K \times \mathbb{R}^N) \quad \text{and} \quad \varphi_n^\perp \rightarrow \varphi^\perp \quad \text{in } L^2(K \times \mathbb{R}^N).$$

Furthermore, using the estimate in (4.8) we get that

$$\sup_{y \in K, x \in \mathbb{R}^N} \left| (1 + |x|)^{N-4} \left[ U_\epsilon\left(\frac{y}{\sqrt{\epsilon}}, x\right) - \mu_0^{-\frac{N-2}{2}}(y) w_0\left(\frac{x - \sqrt{\epsilon}\Phi_1(y)}{\mu_0(y)}\right) \right] \right| \rightarrow 0,$$

as  $\epsilon \rightarrow 0$ , where  $\mu_0$  and  $\Phi_1$  are the smooth explicit function defined in (4.33) and (4.57).

Passing to the limit as  $n \rightarrow \infty$  in (6.42) and applying dominated convergence Theorem, we get

$$\int_{K \times \mathbb{R}^N} \left[ |\nabla_x \varphi^\perp|^2 - p \left( (\mu_0)^{-\frac{N-2}{2}}(y) w_0\left(\frac{x - \sqrt{\epsilon}\Phi_1(y)}{\mu_0(y)}\right) \right)^{p-1} (\varphi^\perp)^2 \right] dy dx \leq 0. \quad (6.43)$$

Furthermore, passing to the limit in the orthogonality condition we get, for any  $y \in K$

$$\int_{\mathbb{R}^N} \varphi^\perp(y, x) Z_0 \left( \frac{x - \sqrt{\epsilon} \Phi_1(y)}{\mu_0(y)} \right) dx = 0, \quad (6.44)$$

$$\int_{\mathbb{R}^N} \varphi^\perp(y, x) Z_j \left( \frac{x - \sqrt{\epsilon} \Phi_1(y)}{\mu_0(y)} \right) dx = 0, \quad j = 1, \dots, N \quad (6.45)$$

and

$$\int_{\mathbb{R}^N} \varphi^\perp(y, x) Z \left( \frac{x - \sqrt{\epsilon} \Phi_1(y)}{\mu_0(y)} \right) dx = 0. \quad (6.46)$$

We thus get a contradiction with (6.43), since for any function  $\varphi^\perp$  satisfying the orthogonality conditions (6.44)–(6.46) for any  $y \in K$  one has

$$\int_{K \times \mathbb{R}^N} \left[ |\nabla_x \varphi^\perp|^2 - p \left( \mu_0^{-\frac{N-2}{2}}(y) w_0 \left( \frac{x - \sqrt{\epsilon} \Phi_1(y)}{\mu_0(y)} \right) \right)^{p-1} (\varphi^\perp)^2 \right] dy dx > 0.$$

**Step 2.** For all  $\epsilon > 0$  small, the functional  $P_\epsilon(\delta)$  defined in (6.16) is continuous and differentiable in  $\mathcal{H}^1(K)$ . Furthermore,  $P_\epsilon$  is a small perturbation in  $(\mathcal{H}^1(K))^N$  of

$$P(\delta) = \frac{A}{2} \epsilon \left[ \int_K |\nabla_K \delta|^2 + a_N \int_K \mathcal{H} \delta^2 \pm \epsilon b_N \int_K \frac{\delta^2}{\mu_0} \right]$$

where  $A = \int_{\mathbb{R}^N} Z_0^2$  and  $\mathcal{H}(y)$  defined in (4.34). Since we are assuming that  $\mu_0$  is a nondegenerate solution to the following problem,

$$-\Delta_K \mu + a_N \mathcal{H}(y) \mu \pm \frac{b_N}{\mu} = 0, \quad \text{in } K,$$

the operator  $P$  is invertible. Thus, for each  $f \in L^2(\Omega_{\epsilon, \gamma})$ ,

$$\delta \in \mathcal{H}^1(K) \longrightarrow \mathbb{R}, \quad \delta \longmapsto P_\epsilon(\delta) - \mathcal{L}_f^3(\delta)$$

has a unique critical point  $\delta$ , which satisfies

$$\epsilon^{-\frac{k}{2}} \|\delta\|_{\mathcal{H}^1(K)} \leq \tilde{\sigma} \epsilon^{-2} \|f\|_{L^2(\Omega_{\epsilon, \gamma})}$$

for some proper  $\tilde{\sigma} > 0$ .

**Step 3.** For all  $\epsilon > 0$  small, the functional  $Q_\epsilon$  defined in (6.17) is a small perturbation in  $(\mathcal{H}^1(K))^N$  of the quadratic form  $\epsilon Q_0(d)$ , defined by

$$\epsilon Q_0(d) = \frac{\epsilon}{2} C \left[ \int_K |\nabla_K d|^2 + \int_K (\tilde{g}^{ab} R_{mabl} - \Gamma_a^c(E_m) \Gamma_c^a(E_l)) d^m d^l \right]$$

with  $C := \int_{\mathbb{R}^N} Z_1^2$  and the terms  $R_{maal}$  and  $\Gamma_a^c(E_m)$  are smooth functions on  $K$ . Recall that the non-degeneracy assumption on the minimal submanifold  $K$  is equivalent to the invertibility of the operator  $Q_0(d)$ .

A consequence, for each  $f \in L^2(\Omega_{\epsilon, \gamma})$ ,

$$d \in (\mathcal{H}^1(K))^N \longrightarrow \mathbb{R}, \quad d \longmapsto Q_\epsilon(d) - \mathcal{L}_f^3(d)$$

has a unique critical point  $d$ , which satisfies

$$\epsilon^{-\frac{k}{2}} \|d\|_{(\mathcal{H}^1(K))^N} \leq \tilde{\sigma} \epsilon^{-2} \|f\|_{L^2(\Omega_{\epsilon, \gamma})}$$

for some proper  $\tilde{\sigma} > 0$ .

**Step 4.** Let  $f \in L^2(\Omega_{\epsilon,\gamma})$  and assume that  $e$  is a given (fixed) function in  $\mathcal{H}^1(K)$ . We claim that for all  $\epsilon > 0$  small enough, the functional  $\mathcal{Q} : (H_\epsilon^1)^\perp \times (\mathcal{H}^1(K))^N \rightarrow \mathbb{R}$

$$(\phi^\perp, \delta, d) \rightarrow \mathcal{E}(\phi^\perp, \delta, d, e)$$

has a critical point  $(\phi^\perp, \delta, d)$ . Furthermore there exists a positive constant  $C$ , independent of  $\epsilon$ , such that

$$\|\phi^\perp\| + \epsilon^{-\frac{k}{2}} \left[ \|\delta\|_{\mathcal{H}^1(K)} + \|d\|_{(\mathcal{H}^1(K))^N} \right] \leq C\epsilon^{-2} \left[ \|f\|_{L^2(\Omega_{\epsilon,\gamma})} + \epsilon^{-\frac{k}{2}} \epsilon^2 \|e\|_{\mathcal{H}^1(K)} \right]. \quad (6.47)$$

To prove the above assertion, we first consider the functional

$$\mathcal{Q}_0(\phi^\perp, \delta, d) = \mathcal{Q}(\phi^\perp, \delta, d, e) - \mathcal{G}(\phi^\perp, \delta, d, e)$$

where  $\mathcal{G}$  is the functional that recollects all mixed terms, as defined in (6.15). A direct consequence of Step 1, Step 2 and Step 3 is that  $\mathcal{Q}_0$  has a critical point  $(\phi^\perp = \phi^\perp(f), \delta = \delta(f), d = d(f))$ , namely the system

$$D_{\phi^\perp} E(\phi^\perp) = D_{\phi^\perp} \mathcal{L}_f^1(\phi^\perp), \quad \epsilon^{-\frac{k}{2}} D_\delta P_\epsilon(\delta) = D_\delta \mathcal{L}_f^2(\delta), \quad \epsilon^{-\frac{k}{2}} D_d Q_\epsilon(d) = D_d \mathcal{L}_f^3(d)$$

is uniquely solvable in  $(H_\epsilon^1)^\perp \times (\mathcal{H}^1(K))^N$  and furthermore

$$\|\phi^\perp\|_{H_\epsilon^1} + \epsilon^{-\frac{k}{2}} \|\delta\|_{\mathcal{H}^1(K)} + \epsilon^{-\frac{k}{2}} \|d\|_{(\mathcal{H}^1(K))^N} \leq C\epsilon^{-2} \|f\|_{L^2(\mathcal{M}_{\epsilon,\gamma})}$$

for some constant  $C > 0$ , independent of  $\epsilon$ .

If we now consider the complete functional  $\mathcal{Q}$ , a critical point of  $\mathcal{Q}x$  shall satisfy the system

$$\begin{cases} D_{\phi^\perp} E(\phi^\perp) = D_{\phi^\perp} \mathcal{L}_f^1(\phi^\perp) + D_{\phi^\perp} \mathcal{G}(\phi^\perp, \delta, d, e) \\ D_\delta P_\epsilon(\delta) = D_\delta \mathcal{L}_f^2(\delta) + D_\delta \mathcal{G}(\phi^\perp, \delta, d, e) \\ D_d Q_\epsilon(d) = D_d \mathcal{L}_f^3(d) + D_d \mathcal{G}(\phi^\perp, \delta, d, e). \end{cases} \quad (6.48)$$

On the other hand, as we have already observed in Theorem 6.1, we have

$$\|D_{(\phi^\perp, \delta, d)} \mathcal{G}(\phi_1^\perp, \delta_1, d_1, e_1) - D_{(\phi^\perp, \delta, d)} \mathcal{G}(\phi_2^\perp, \delta_2, d_2, e_2)\| \leq C\epsilon^2 \times$$

$$\left[ \|\phi_1^\perp - \phi_2^\perp\| + \epsilon^{-\frac{k}{2}} \|\delta_1 - \delta_2\|_{\mathcal{H}^1(K)} + \epsilon^{-\frac{k}{2}} \|d_1 - d_2\|_{(\mathcal{H}^1(K))^N} + \epsilon^{-\frac{k}{2}} \|e_1 - e_2\|_{\mathcal{H}^1(K)} \right].$$

Thus the contraction mapping Theorem guarantees the existence of a unique solution  $(\bar{\phi}^\perp, \bar{\delta}, \bar{d})$  to (6.48) in the set

$$\|\phi^\perp\|_{H_\epsilon^1} + \epsilon^{-\frac{k}{2}} \|\delta\|_{\mathcal{H}^1(K)} + \epsilon^{-\frac{k}{2}} \|d\|_{(\mathcal{H}^1(K))^N} \leq C \left[ \epsilon^{-2} \|f\|_{L^2(\Omega_{\epsilon,\gamma})} + \epsilon^2 \epsilon^{-\frac{k}{2}} \|e\|_{\mathcal{H}^1(K)} \right].$$

Furthermore, the solution  $\bar{\phi}^\perp = \bar{\phi}^\perp(f, e)$ ,  $\bar{\delta} = \bar{\delta}(f, e)$  and  $\bar{d} = \bar{d}(f, e)$  depend on  $e$  in a smooth and non-local way.

**Step 5.** Given  $f \in L^2(\Omega_{\epsilon,\gamma})$ , we replace the critical point  $(\bar{\phi}^\perp = \bar{\phi}^\perp(f, e), \bar{\delta} = \bar{\delta}(f, e), \bar{d} = \bar{d}(f, e))$  of  $\mathcal{Q}$  obtained in the previous step into the functional  $\mathcal{E}(\phi^\perp, \delta, d, e)$  thus getting a new functional depending only on  $e \in \mathcal{H}^1(K)$ , that we denote by  $\mathcal{F}_\epsilon(e)$ , given by

$$\begin{aligned} \mathcal{F}_\epsilon(e) &= \epsilon^{-\frac{k}{2}} [R_\epsilon(e) - \mathcal{L}_f^4(e)] + E(\bar{\phi}^\perp(e)) - \epsilon^{-\frac{k}{2}} \mathcal{L}_f^1(\bar{\phi}^\perp(e)) + \epsilon^{-\frac{k}{2}} [P_\epsilon(\bar{\delta}(e)) - \mathcal{L}_f^2(\bar{\delta}(e))] \\ &+ \epsilon^{-\frac{k}{2}} [Q_\epsilon(\bar{d}(e)) - \mathcal{L}_f^3(\bar{d}(e))] + \mathcal{G}(\bar{\phi}^\perp(e), \bar{\delta}(e), \bar{d}(e), e). \end{aligned}$$

The rest of the proof is devoted to show that there exists a sequence  $\epsilon = \epsilon_l \rightarrow 0$  such that

$$D_e \mathcal{F}_\epsilon(e) = 0 \quad (6.49)$$

is solvable. Using the fact that  $(\bar{\phi}^\perp, \bar{\delta}, \bar{d})$  is a critical point for  $\mathcal{Q}$  (see Step 4 for the definition), we have that

$$D_e \mathcal{F}_\epsilon(e) = \epsilon^{-\frac{k}{2}} D_e [R_\epsilon(e) - \mathcal{L}_f^4(e)] + D_e \mathcal{G}(\bar{\phi}^\perp(e), \bar{\delta}(e), \bar{d}(e), e). \quad (6.50)$$

Define

$$\mathcal{L}_\epsilon := \epsilon^{-\frac{k}{2}} D_e R_\epsilon(e) + D_e \mathcal{G}(\bar{\phi}^\perp(e), \bar{\delta}(e), \bar{d}(e), e), \quad (6.51)$$

regarded as self adjoint in  $\mathcal{L}^2(K)$ . The work to solve the equation  $D_e \mathcal{F}_\epsilon(e) = 0$  consists in showing the existence of a sequence  $\epsilon_l \rightarrow 0$  such that 0 lies suitably far away from the spectrum of  $\mathcal{L}_{\epsilon_l}$ .

We recall now that the map

$$(\phi^\perp, \delta, d, e) \rightarrow D_e \mathcal{G}(\phi^\perp, \delta, d, e)$$

is a linear operator in the variables  $\phi^\perp, \delta, d$ , while it is constant in  $e$ . This is contained in the result of Theorem 6.1. If we furthermore take into account that the terms  $\bar{\phi}^\perp, \bar{\delta}$  and  $\bar{d}$  depend smoothly and in a non-local way through  $e$ , we conclude that, for any  $e \in \mathcal{H}^1(K)$ ,

$$D_e \mathcal{G}(\bar{\phi}^\perp(e), \bar{\delta}(e), \bar{d}(e), e)[e] = \epsilon^{\gamma(N-3)} \epsilon^{-\frac{k}{2}} \int_K (\epsilon \eta_1(e) \partial_a e + \eta_2(e) e)^2 \quad (6.52)$$

where  $\eta_1$  and  $\eta_2$  are non local operators in  $e$ , that are bounded, as  $\epsilon \rightarrow 0$ , on bounded sets of  $\mathcal{L}^2(K)$ . Thanks to the result contained in Theorem 6.1 and the above observation, we conclude that the quadratic form

$$\Upsilon_\epsilon(e) := \epsilon^{-\frac{k}{2}} D_e R_\epsilon(e)[e] + D_e \mathcal{G}(\bar{\phi}^\perp(e), \bar{\delta}(e), \bar{d}(e), e)[e]$$

can be described as follows

$$\tilde{\Upsilon}_\epsilon(e) = \epsilon^{\frac{k}{2}} \Upsilon_\epsilon(e) = \Upsilon_\epsilon^0(e) - \bar{\lambda}_0 \int_K e^2 + \epsilon \Upsilon_\epsilon^1(e) \quad (6.53)$$

where

$$\Upsilon_\epsilon^0(e) = \epsilon^2 \int_K (1 + \epsilon^{\gamma(N-3)} \eta_1(e)) \left| \partial_a \left( e(1 + e^{-\epsilon^{-\lambda'}} \beta_3^\epsilon(y)) \right) \right|^2. \quad (6.54)$$

In the above expression  $\bar{\lambda}_0$  is the positive number defined by

$$\bar{\lambda}_0 = \left( \int_{\mathbb{R}^N} Z_1^2 \right) \lambda_0,$$

$\Upsilon_\epsilon^1(e)$  is a compact quadratic form in  $\mathcal{H}^1(K)$ ,  $\beta_3^\epsilon$  is a smooth and bounded (as  $\epsilon \rightarrow 0$ ) function on  $K$ , given by (6.20). Finally,  $\eta_1$  is a non local operator in  $e$ , which is uniformly bounded, as  $\epsilon \rightarrow 0$  on bounded sets of  $\mathcal{L}^2(K)$ .

Thus, for any  $\epsilon > 0$ , the eigenvalues of

$$\mathcal{L}_\epsilon e = \lambda e, \quad e \in \mathcal{H}^1(K)$$

are given by a sequence  $\lambda_j(\epsilon)$ , characterized by the Courant-Fisher formulas

$$\lambda_j(\epsilon) = \sup_{\dim(M)=j-1} \inf_{e \in M^\perp \setminus \{0\}} \frac{\tilde{\Upsilon}_\epsilon(e)}{\int_K e^2} = \inf_{\dim(M)=j} \sup_{e \in M \setminus \{0\}} \frac{\tilde{\Upsilon}_\epsilon(e)}{\int_K e^2}. \quad (6.55)$$

The proof of Theorem 5.1 and of the inequality (5.10) will follow then from Step 4 and formula (6.47), together with the validity of the following

**Lemma 6.2.** *There exist a sequence  $\epsilon_l \rightarrow 0$  and a constant  $c > 0$  such that, for all  $j$ , we have*

$$|\lambda_j(\epsilon_l)| \geq c \epsilon_l^k. \quad (6.56)$$

For the proof of Lemma we refer to [6].

□

## 7. PROOF OF PROPOSITION 4.2

The proof of this Proposition will be divided into several steps.

**Step 1.** Let us assume that  $\phi$  solves (4.16). We claim that there exists  $C > 0$  such that

$$\|\phi\|_{\epsilon, r-2} \leq C \|h\|_{\epsilon, r}. \quad (7.1)$$

By contradiction, assume that there exist sequences  $\epsilon_n \rightarrow 0$ ,  $h_n$  with  $\|h_n\|_{\epsilon_n, r} \rightarrow 0$  and solutions  $\phi_n$  to (4.16) with  $\|\phi_n\|_{\epsilon_n, r-2} = 1$ .

Let  $z_n \in K_{\epsilon_n}$  and  $\xi_n$  be such that

$$|\phi_n(\epsilon_n z_n, \xi_n)| = \sup |\phi_n(y, \xi)|.$$

We may assume that, up to subsequences,  $(\epsilon_n z_n) \rightarrow \bar{y}$  in  $K$ . Furthermore, we have by assumption that  $|\xi_n| \leq \eta \epsilon_n^{-\frac{1}{2}}$ .

Let us now assume that there exists a positive constant  $R$  such that  $|\xi_n| \leq R$ . In this case, up to subsequences, one gets that  $\xi_n \rightarrow \xi_0$ . Consider the functions

$$\tilde{\phi}_n(z, \xi) = \phi_n(z, \xi + \xi_n), \quad \text{for } (z, \xi) \in K_{\epsilon_n} \times \{\xi \in \mathbb{R}^N : |\xi| \leq \eta' \epsilon_n^{-\frac{1}{2}}\}$$

for some  $\eta' > 0$ . This is a sequence of uniformly bounded functions, that converges uniformly over compact sets of  $K \times \mathbb{R}^N$  to a function  $\tilde{\phi}$  solution to

$$-\Delta \tilde{\phi} - p w_0^{p-1} \tilde{\phi} = 0 \quad \text{in } \mathbb{R}^N$$

Since the orthogonality conditions pass to the limit, we get that furthermore

$$\int_{\mathbb{R}^N} \tilde{\phi}(y, \xi) Z_j(\xi) d\xi = 0 \quad \text{for all } y \in K, \quad j = 0, \dots, N.$$

These facts imply that  $\tilde{\phi} \equiv 0$ , that is a contradiction.

Assume now that  $\lim_{n \rightarrow \infty} |\xi_n| = \infty$ . Consider the scaled function

$$\tilde{\phi}_n(z, \xi) = \phi_n(z, |\xi_n| \xi + \xi_n)$$

defined on the set

$$\tilde{\mathcal{D}} = \left\{ (z, \xi) : z \in K_{\epsilon_n}, |\xi| < \frac{\eta}{\sqrt{\epsilon_n} |\xi_n|} - \frac{\xi_n}{|\xi_n|}, \right\}.$$

Thus  $\tilde{\phi}_n$  satisfies the equation

$$\Delta \tilde{\phi}_n + p C_N \frac{|\xi_n|^2}{(1 + |\xi_n| \xi + \xi_n|^2)^2} \tilde{\phi}_n - |\xi_n|^2 \epsilon_n a \tilde{\phi}_n = |\xi_n|^2 h(z, |\xi_n| \xi + \xi_n) \quad \text{in } \tilde{\mathcal{D}}.$$

Consider first the case in which  $\lim_{n \rightarrow \infty} \epsilon_n |\xi_n|^2 = 0$ . Under our assumptions, we have that  $\tilde{\phi}_n$  is uniformly bounded and it converges locally over compact sets to  $\tilde{\phi}$  solution to

$$\Delta \tilde{\phi} = 0, \quad |\tilde{\phi}| \leq C |\xi|^{2-r} \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since  $4 < r < N$ , we conclude that  $\tilde{\phi} \equiv 0$ , which is a contradiction.

Consider now the other possible case, namely that

$$\lim_{n \rightarrow \infty} \epsilon_n |\xi_n|^2 = \beta > 0.$$

Then,

$$\tilde{\mathcal{D}} \rightarrow \mathcal{S} := \{\xi : |\xi| \in [0, L)\} \quad \text{as } n \rightarrow \infty$$

where  $L$  is some positive constant. Furthermore, up to subsequences, we get that  $\tilde{\phi}_n$  converges uniformly over compact sets to  $\tilde{\phi}$  solution to

$$\Delta \tilde{\phi} - \beta a \tilde{\phi} = 0, \quad |\tilde{\phi}| \leq C |\xi|^{2-r} \quad \text{in } \mathcal{S}, \quad \tilde{\phi} = 0 \quad \text{on } \partial \mathcal{S}.$$



Multiplying equation by  $\tilde{\phi}$ , and integrating it over  $\mathcal{S}$  only in  $\xi$ , we get

$$\int_{\mathcal{S}} (|\nabla \tilde{\phi}|^2 + \beta a \tilde{\phi}^2) d\xi = 0.$$

Thus we conclude that  $\tilde{\phi} \equiv 0$ , which is a contradiction. The proof of (7.1) is completed.

**Step 2.** We shall now show that there exists  $C > 0$  such that, if  $\phi$  is a solution to (4.16), then

$$\|D_{\xi}^2 \phi\|_{\epsilon, r} + \|D_{\xi} \phi\|_{\epsilon, r-1} + \|\phi\|_{\epsilon, r-2} \leq C \|h\|_{\epsilon, r}. \quad (7.2)$$

For  $z \in K_{\epsilon}$ , we have that  $\phi$  solves  $-\Delta \phi = \tilde{h}$  in  $|\xi| < \eta \epsilon^{-\frac{1}{2}}$  where  $|\tilde{h}| \leq \frac{C}{(1+|\xi|^r)}$ , for some constant  $C > 0$ . Elliptic estimates give that  $|\phi| \leq \frac{C}{(1+|\xi|^{r-2})}$ .

Let us now fix a point  $e \in \mathbb{R}^N$  and a positive number  $R > 0$ . Perform the change of variables  $\tilde{\phi}(z, t) = \phi(z, Rt + 3Re)$ , so that

$$\Delta \tilde{\phi} = \frac{1}{R^{r-2}} \tilde{h} \quad \text{in } |t| \leq 1$$

where  $|\tilde{h}| \leq \frac{C}{|t+3e|^r}$ . Elliptic estimates give then that  $\|R^{r-2} D^2 \tilde{\phi}\|_{L^{\infty}(0,1)} \leq C \|\tilde{h}\|_{L^{\infty}(B(0,2))}$ , inequality that translates into

$$\|R^r D^2 \phi\|_{L^{\infty}(B(3Re, R))} \leq C \|(1+|\xi|)^r h\|_{L^{\infty}(|\xi| \leq \eta \epsilon^{-\frac{1}{2}})}.$$

This inequality finally gives

$$\|(1+|\xi|)^r D^2 \phi\|_{L^{\infty}(|\xi| \leq \eta \epsilon^{-\frac{1}{2}})} \leq C \|(1+|\xi|)^r h\|_{L^{\infty}(|\xi| \leq \eta \epsilon^{-\frac{1}{2}})}.$$

Arguing in a similar way, one gets the internal weighted estimate for the first derivative of  $\phi$

$$\|(1+|\xi|)^{r-1} D \phi\|_{L^{\infty}(|\xi| \leq \eta \epsilon^{-\frac{1}{2}})} \leq C \|(1+|\xi|)^r h\|_{L^{\infty}(|\xi| \leq \eta \epsilon^{-\frac{1}{2}})}.$$

By using the representation formula for solution  $\phi$  to the above equation, we see that  $|\phi| \leq C \epsilon^{\frac{r-2}{2}}$  in  $|\xi| < \eta \epsilon^{-\frac{1}{2}}$ . Furthermore, elliptic estimates give that in this region  $|D \phi| \leq C \epsilon^{\frac{r-1}{2}}$  and  $|D^2 \phi| \leq C \epsilon^{\frac{r}{2}}$ . This concludes the proof of (7.2).

**Step 3.** We shall now show that there exists  $C > 0$  such that, if  $\phi$  is a solution to (4.16), then

$$\|D_{\xi}^2 \phi\|_{\epsilon, r, \sigma} + \|D_{\xi} \phi\|_{\epsilon, r-1, \sigma} + \|\phi\|_{\epsilon, r-2, \sigma} \leq C \|h\|_{\epsilon, r, \sigma}. \quad (7.3)$$

From elliptic regularity, we have that if  $\|h\|_{\epsilon, r, \sigma} \leq C$  then  $\|\phi\|_{\epsilon, r-2, \sigma} \leq C$ . Thus, we write that  $\phi$  solves  $-\Delta \phi = \tilde{h}$  in  $|\xi| < \eta \epsilon^{-\frac{1}{2}}$  where  $\|\tilde{h}\|_{\epsilon, r, \sigma} \leq C$ .

Arguing as in the previous step, we fix a point  $e \in \mathbb{R}^N$  and a positive number  $R > 0$ . Perform the change of variables  $\tilde{\phi}(z, t) = \phi(z, Rt + 3Re)$ , so that

$$\Delta \tilde{\phi} = \frac{1}{R^{r-2}} \tilde{h} \quad \text{in } |t| \leq 1$$

where  $|\tilde{h}| \leq \frac{C}{|t+3e|^r}$ . Elliptic estimates give then that  $\|R^{r-2} D^2 \tilde{\phi}\|_{C^{0, \sigma}(B(0,1))} \leq C \|\tilde{h}\|_{L^{\infty}(B(0,2))}$ . This implies that

$$R^{r-2} \|D_{\xi}^2 \tilde{\phi}\|_{L^{\infty}(B_1)} + R^{r-2} \|D^2 \tilde{\phi}\|_{\sigma, B(0,1)} \leq C.$$

In particular, we have for any  $z \in K_{\epsilon}$ , that

$$R^{r-2} \sup_{y_1, y_2 \in B(0,1)} \frac{|D^2 \tilde{\phi}(z, y_1) - D^2 \tilde{\phi}(z, y_2)|}{|y_1 - y_2|^{\sigma}} \leq C.$$

This inequality gets translated in term of  $\phi$  as

$$R^{r+\sigma} \sup_{\xi_1, \xi_2 \in B(\xi, 1)} \frac{|D^2 \phi(z, \xi_1) - D^2 \phi(z, \xi_2)|}{|\xi_1 - \xi_2|^{\sigma}} \leq C.$$

In a very similar way, one gets the estimate on  $D\phi$ . This concludes the proof of (7.3).

**Step 4.** Differentiating equation (4.16) with respect to the  $z$  variable  $l$  times and using elliptic regularity estimates, one proves that

$$\|D_y^l \phi\|_{\epsilon, r-2, \sigma} \leq C_l \left( \sum_{k \leq l} \|D_y^k h\|_{\epsilon, r, \sigma} \right) \quad (7.4)$$

for any given integer  $l$ .

**Step 5.** Now we shall prove the existence of the solution  $\phi$  to problem (4.16). We consider the Hilbert space  $\mathcal{H}$  defined as the subspace of functions  $\psi$  which are in  $H^1(\mathcal{D})$  such that  $\psi = 0$  on  $\partial\hat{\mathcal{D}}$ , and

$$\int_{\hat{\mathcal{D}}} \psi(\epsilon z, \xi) Z_j(\xi) d\xi = 0 \text{ for all } z \in K_\epsilon, \quad j = 0, \dots, N.$$

Define a bilinear form in  $\mathcal{H}$  by

$$B(\phi, \psi) := \int_{\hat{\mathcal{D}}} \psi L\phi.$$

Then problem (4.16) gets weakly formulated as that of finding  $\phi \in \mathcal{H}$  such that

$$B(\phi, \psi) = \int_{\hat{\mathcal{D}}} h\psi \quad \forall \psi \in \mathcal{H}.$$

By the Riesz representation theorem, this is equivalent to solve

$$\phi = T(\phi) + \tilde{h}$$

with  $\tilde{h} \in \mathcal{H}$  depending linearly on  $h$ , and  $T : \mathcal{H} \rightarrow \mathcal{H}$  being a compact operator. Fredholm's alternative guarantees that there is a unique solution to problem (4.16) for any  $h$  provided that

$$\phi = T(\phi) \quad (7.5)$$

has only the zero solution in  $\mathcal{H}$ . Equation (7.5) is equivalent to problem (4.16) with  $h = 0$ . If  $h = 0$ , the estimate in (4.17) implies that  $\phi = 0$ .

This concludes the proof of Proposition 4.2.

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